THE UNIQUE EXISTENTIAL QUANTIFIER*

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In elementary number theory a set is definable by a formula

\( (\exists x_1) \cdots (\exists x_n) \varphi \)

(where \( \varphi \) defines a recursive relation) if and only if it is the intersection of a \( \Pi_n \) set and a \( \Sigma_n \) set. This was proved by Rödding, [6, Satz 1, p. 62]. In the present paper we investigate the extent to which analogous results hold in second-order number theory.

The unique existential quantifier \( (\exists! x) \) is interpreted as meaning "there is a unique function \( x \) such that." Let \( (\exists! x) \) be the class of relations which are definable by formulas \( (\exists! x) \) formulas

\( (\exists! x_1) \cdots (\exists! x_n) \varphi \)

where \( \varphi \) defines an arithmetical relation. Here \( \varphi \) may contain free number variables or function variables or both. Let \( (\exists! x) \) be the class of relations which are the intersection of a \( \Pi_n \) relation with a \( \Sigma_n \) relation. (Or in other words, which are the difference of two \( \Pi_n \) relations.) A relationship in one direction between these two notions is easy to establish:

**Theorem 1:** \( (\exists! x) \subseteq (\exists! x) \).

**Proof:** Use induction on \( n \). \( (\exists! x) \varphi(x) \) is equivalent to

\[ (\exists x \varphi(x) \& \forall x \forall x' (\varphi(x) \& \varphi(x') \rightarrow x = x') \).

If \( \varphi \) defines a \( \Pi_n \cdot \Sigma_n \) (and hence \( \Delta_n^{1+1} \)) relation then the above defines a \( \Pi_n^{1+1} \cdot \Sigma_n^{1+1} \) relation.

The upper bound on \( (\exists! x) \) can be improved in the \( n = 1 \) case; \( (\exists! x) \subseteq (\exists! x) \). This is because any function which is the unique solution to an arithmetical condition must itself be hyperarithmetical. Thus if \( \varphi \) defines an arithmetical relation then

\[ (\exists x \text{ hyperarithmetical in } \gamma) \varphi(x, \gamma) \& \forall x \forall x' [\varphi(x, \gamma) \& \varphi(x', \gamma) \rightarrow x = x'] \).

The above defines a \( \Pi_1 \) relation by well-known results of Kleene [3, p. 26]. For lower bounds on \( (\exists! x) \), a basic tool is the following validity:

\[ \forall x \varphi(x) \leftrightarrow (\exists! x) (\varphi(\lambda t 0) \& (\neg \varphi(x) \lor x = \lambda t 0)) \]

\* Eingegangen am 13. 1. 69.
The Unique Existential Quantifier

(This is analogous to Rödding's \( (a_3) \), [6, p. 62]. We could equally well use the version employed by Lusin [4, p. 259]

\[ \forall x \varphi(x) \leftrightarrow (\exists! x) \left( \neg \varphi(\lambda t x(t) = 1) \lor x = \lambda t 0 \right). \]

By using (*) and the preceding paragraph we have:

**Theorem 2:**

\[ \exists!_1^1 = \Pi_1^1. \]

This result is reasonably well known; it is the effective version of a result in descriptive set theory (see Lusin [4, p. 259]).

By use of the Kondo-Addison uniformization theorem [8, p. 188] we next prove:

**Theorem 3:**

\[ \exists!_{1/2}^1 = \Pi_2^1 \cdot \Sigma_2^1. \]

**Proof:** It is easy to see that the intersection of two relations in \( \exists!_{1/2}^1 \) is again in \( \exists!_{1/2}^1 \).

(This is because two sets are both singletons if and only if their cartesian product is a singleton.) So it suffices to show that \( \Pi_2^1 \subseteq \exists!_{1/2}^1 \) and \( \Sigma_2^1 \subseteq \exists!_{1/2}^1 \). Suppose \( \varphi \) defines a \( \Pi_1^1 \) set, and \( \varphi' \) defines a \( \Pi_1^1 \) set which uniformizes it. Then

\[ \exists x \varphi, \exists x \varphi', \text{ and } (\exists! x) \varphi' \]

are equivalent. By applying Theorem 2 to \( \varphi' \), we get \( \Sigma_2^1 \subseteq \exists!_{1/2}^1 \).

For the other half, consider a formula \( \psi \) defining a \( \Sigma_1^1 \) relation. As in (*), \( \psi(x) \) is equivalent to

\[ \psi(\lambda t 0) \& (\exists! x) \left[ \neg \psi(x) \lor x = \lambda t 0 \right]. \]

The first conjunct is \( \Sigma_1^1 \) and hence \( \Sigma_2^1 \) and hence \( \exists!_{1/2}^1 \). The part in square brackets is essentially \( \Pi_1^1 \) and hence \( \exists!_{1/2}^1 \). Hence the entire formula is essentially \( \exists!_{1/2}^1 \).

By applying (*) to the \( \Sigma_2^1 \) half of Theorem 3 we obtain at once:

**Corollary 4:**

\[ \Pi_{1/2}^1 \subseteq \exists!_{1/2}^1 \subseteq \Pi_1^1 \cdot \Sigma_2^1. \]

Theorem 3 can be extended if we have stronger uniformization principles available.

**Theorem 5:** Assume that \( \Pi_n^1 \) relations can be uniformized by \( \Pi_{n+1}^1 \cdot \Sigma_n^1 \) relations, for all \( n \geq 1 \). Then for all \( n \geq 2 \),

\[ \exists!_{1/n}^1 = \Pi_n^1 \cdot \Sigma_n^1. \]

**Proof:** Use induction on \( n \). That \( \Pi_{n+1}^1 \subseteq \exists!_{1/n}^1 \) follows directly from (*) and the inductive hypothesis. Next consider the \( \Sigma_{n+1}^1 \) relation defined by \( \exists x \varphi \). Uniformize the \( \Pi_n^1 \) relation defined by \( \varphi \) by a relation definable by the \( \exists!_{1/n}^1 \) formula \( \varphi \).

Then \( \exists x \varphi \) is equivalent to \( (\exists! x) \varphi \).

The hypothesis of Theorem 5 is a consequence of the axiom of constructibility, see Addison [1]. Hence:

**Corollary 6:** The axiom of constructibility implies that

\[ \exists!_{1/n}^1 = \Pi_n^1 \cdot \Sigma_n^1 \]

for \( n \geq 2 \).