Stabilization of Wave Equations Coupled in Parallel by Viscous Damping

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Abstract. In this article, we consider stabilization of two wave equations coupled in parallel by viscous damping. Uniform exponential stabilization can be achieved when damping is presented in two equations. In general only strong stabilization can be achieved when damping is present in one equation, while uniform exponential stabilization is still possible for some special cases.

1. Introduction

In this article, we wish to consider the stabilization of two wave equations coupled in parallel by viscous damping,

\[ u_{tt} - c_1^2 \Delta u + k_1 u_t = L(v - u), \quad x \in \Omega_1, \ t > 0, \]  
\[ v_{tt} - c_2^2 \Delta v + k_2 v_t = L(u - v), \quad x \in \Omega_2, \ t > 0, \]  

where \( x = (x_1, x_2, \ldots, x_n) \); \( \Omega_1, \Omega_2 \) are bounded, connected, open sets in \( R^n \) with piecewise smooth boundary; \( k_1 \geq 0, k_2 \geq 0 \) are damping constants; and \( L > 0 \) is the coupling constant.

The open set \( \Omega_1 \) is on the hyperplane \( x_{n+1} = 0 \), and \( \Omega_2 \) is on the hyperplane \( x_{n+1} = h \) in \( R^{n+1} \). We assume that the projection of \( \Omega_2 \) into the hyperplane \( X_{n+1} = 0 \), denoted as \( \Omega \), is \( \Omega_1 \). Dirichlet boundary conditions are employed at the boundary

\[ u(x, t)\big|_{x \in \partial \Omega} = 0, \quad v(x, t)\big|_{x \in \partial \Omega} = 0. \]  

\( u(x, t) \) and \( v(x, t) \) are the displacement of two vibrating objects measured from their equilibrium positions, respectively. For illustration purpose, a problem in \( R^1 \) can be depicted as in figure 1, where springs should be perceived as distributed ones.

The coupling terms \( L(v - u) \) and \( L(u - v) \) can be regarded as representing a distributed spring linking two vibrating objects. Energy can flow from one object to another via linkage. This problem is motivated by an analogous problem in ordinary differential equations for coupled oscillators, and has potential application in isolation of objects from outside disturbances.
Stabilization properties of serially connected vibrating strings or beams can be found in several papers (see [3, 4], for example). In both cases, uniform stabilization can be achieved if dissipative boundary conditions are enforced at one endpoint. If one damper is installed at the midspan joint of two coupled vibrating strings, the uniform stabilization property still holds if the ratio of the wave speeds in the two strings, $c_1/c_2$, has certain rational values ($1, 4/5$ for example), while there are undamped solutions if $c_1/c_2$ has other rational values ($4/5$, for example). Stabilization properties of vibrating strings connected in parallel also show some dependence upon the system parameters. The main purpose of this article is to study the uniform or strong stabilization properties of equations (1) to (3) under different viscous damping in the region. Except in theorem 3, $k_1$ and $k_2$ are assumed to be constants. The case where $k_1$ and $k_2$, defined on $\Omega$, are positive functions of $x$, uniformly bounded away from zero, can be treated with little or no change in the arguments.

The outline of the article is as follows. In section 2 we set out our notation, reformulate the system (1)-(3) into an evolution system, and give preliminary results on well-posedness of the basic problem. It is well known that solutions of a wave equation with viscous damping decays to zero uniformly exponentially. In section 3 we first prove that the same property also holds for the system (1)-(3) if $k_1 > 0$ and $k_2 > 0$ by using the energy perturbation method. It is natural to ask whether uniform exponential decay still holds when one of the $k_i$, $i = 1, 2$, is zero. An example given in this section shows that this is not true when $c_1 \neq c_2$ in a parallelepipedon in $R^n$. The complete answer to the question is still open, and needs further investigation. Finally, using a theorem in [2] concerning exponential decay of evolution equations with locally distributed damping, we are able to prove a result for wave equations in a rectangular region in $R^2$ with locally distributed damping: all the solutions of system (1)-(3) decay to zero uniformly exponentially if $c_1 = c_2$, $k_2 = 0$, and there exists an open set $\Omega_0 \subseteq \Omega$ such that $\inf\{k_1(x) \mid x \in \Omega_0\} \geq k_0 > 0$, where $\Omega_0$ contains...