A general treatment of two dimensional plane flows for a micropolar fluid

By P. L. Bhatnagar and Renuka Ravindran

§ 1. It is the aim of this paper to derive a unified treatment for two-dimensional plane flow for a micropolar fluid and to study the underlying common behaviour exhibited by the fluid in these flows. An earlier investigation was done for Rivlin-Ericksen fluids in two-dimensional plane flows and axisymmetric cyclindrical flows (1).

The components of the velocity and micro-rotation vectors in two-dimensional plane flow have the form:

\[ u = u(x, y, t), \quad v = v(x, y, t), \quad w = 0 \]

\[ v_x = 0, \quad v_y = 0, \quad v_z = v(x, y, t) \]

as a motion in the \( x-y \) plane would sustain a non-zero micro-rotation about the \( z \) axis alone.

The equations governing the motion are equation of continuity:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  

equation of balance of momentum:

\[ -(\mu + K_v) \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right) + K_v \frac{\partial v}{\partial y} \frac{\partial p}{\partial x} + \varphi F_x = \varphi \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right), \]  

\[ -(\mu + K_v) \left( \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} \right) - K_v \frac{\partial v}{\partial x} \frac{\partial p}{\partial y} + \varphi F_y = \varphi \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right). \]  

Equation of balance of first stress moments

\[ \gamma v \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + K_v \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - 2K_v v = \eta \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right), \]  

where \( F_x, F_y \) are assumed to be derived from a potential \( \Omega \), so that

\[ F_x = -\frac{\partial \Omega}{\partial x}, \quad F_y = -\frac{\partial \Omega}{\partial y}. \]  

The condition for integrability for this set of equation is that

\[ \frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}, \]  

from [1.3] and [1.4]. Namely

\[ -(\mu + K_v) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + K_v \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = -\eta \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right), \]

\[ \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \]  

We introduce a stream function \( \psi(x, y) \) by the relation

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \]  

to satisfy the equation of continuity identically. This gives

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \omega = -\Delta \psi \]  

where \( \omega \) is the vorticity present in the fluid. \( \psi \) and \( \nu \) are determined from the equations

\[ (\mu + K_v) \Delta \psi = \eta \frac{D}{Dt} \Delta \psi \]  

and

\[ \gamma_v \Delta \nu - K_v \Delta \psi - 2K_v v = \eta \frac{D}{Dt} \Delta \psi \]  

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From the Department of Applied Mathematics, Indian Institute of Science, Bangalore-12 (India)
\( \omega \) and \( \nu \) satisfy the equation
\[
\psi \frac{D\omega}{Dt} = (\mu_e + K_e) A\omega - K_e A\nu
\]
and
\[
\psi \frac{D\nu}{Dt} = \gamma_e A\nu + K_e A\omega - 2 K_e \nu
\]
[1.12]

together with suitable conditions on the boundary. It is difficult to uncouple these equations in the most general case, so we shall restrict ourselves to flows in which Stokes' approximation is valid (namely, either the boundaries perform small amplitude oscillations or small angular velocity rotations or the perturbation due to the introduction of the boundaries in an uniform stream is small).

\section{2. The equations governing the Stokes flow}

The equations governing the Stokes flow are
\[
(\mu_e + K_e) A\nabla \psi + K_e A\psi = \nabla \frac{\partial}{\partial t} A\psi,
\]
[2.1]
\[
\gamma_e A\nu - K_e A\nabla \psi - 2 K_e \nu = \frac{\partial}{\partial t} A\psi.
\]
[2.2]

From [2.1]
\[
A \left[ (\mu_e + K_e) A\psi + K_e A\nu - \nabla \frac{\partial}{\partial t} A\psi \right] = 0
\]
\[
(\mu_e + K_e) A\nabla \psi + K_e A\nu - \nabla \frac{\partial}{\partial t} A\psi = H(x, y) c(t),
\]
[2.3]
where \( H(x, y) \) is a solution of the harmonic equation and \( c(t) \) is any function of time.

\[
\nu = \frac{1}{K_e} \left[ \frac{\partial}{\partial t} - (\mu_e + K_e) A\nu + H(x, y) c(t) \right],
\]
[2.4]

and \( \psi \) satisfies the equation:
\[
\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \left[ \frac{j(\mu_e + K_e) + \gamma_e \nu}{K_e} \right] \frac{\partial}{\partial t} A\psi + 2 \nabla \frac{\partial}{\partial t} A\psi + \frac{\partial}{\partial t} A\psi = \frac{\partial}{\partial t} A\psi
\]
\[
+ \frac{\gamma_e (\mu_e + K_e)}{K_e} A\nabla \psi - (2 \mu_e + K_e) A\psi = -2H(x, y) c(t)
\]
\[
- \frac{\partial}{\partial \nu} H(x, y) c(t),
\]
[2.5]

together with suitable boundary conditions.

\textbf{Case a:} Exponential or sinusoidal dependence on time. Choose \( c(t) \) of the same form as the dependence of the flow variables on time i.e. \( e^{kt} \) (\( k = -n \) or \( i \alpha \)). This allows us to write \( \psi = \psi_0 e^{kt} \), where
\[
A\psi_0 - a A\psi_0 + b \psi_0 = c H(x, y)
\]
[2.6]

and
\[
a = \frac{\mu_e}{\mu_e + K_e} + \frac{\gamma_e \nu}{\gamma_e (\mu_e + K_e)} - \frac{2 \mu_e + K_e}{\gamma_e (\mu_e + K_e)} \frac{2}{K_e},
\]
\[
b = \frac{k \nu}{\gamma_e (\mu_e + K_e)} \left( 2 + \frac{\mu_e}{K_e} \right),
\]
\[
c = -\frac{\nu}{\gamma_e (\mu_e + K_e)} \left( 2 + \frac{\mu_e}{K_e} \right)
\]
[2.7]

\( \psi_0 \) satisfies the equation
\[
(\lambda + h_1) (\lambda + h_2) \psi_0 = c H(x, y).
\]
[2.8]

Therefore
\[
\psi_0 = \psi_1 + \frac{1}{h_1 - h_2} \psi_1 + \frac{c}{h_1 h_2} H(x, y),
\]
[2.9]

where
\( \psi_1 \) is a solution of \((\lambda + h_1) \psi = 0\)
\( \psi_1 \) is a solution of \((\lambda + h_2) \psi = 0\)

and \( \psi_0 \) satisfies required boundary conditions together with \( \psi_0 \), which is given by
\[
\psi = \frac{1}{K_e} [H(x, y) + \nu \frac{\partial}{\partial t} - (\mu_e + K_e) A\psi_0].
\]
[2.10]

The above expressions can be compared with those obtained by us when studying flow in a channel induced by a pulsating pressure gradient (2). [2.9] and [2.10] can be used to study pulsating flow in a wavy channel or between two visco-elastic membranes.

\textbf{Case b: Steady motion}

The stream function \( \psi \) now satisfies the eq. (cf. [2.5]):
\[
A\psi - \frac{2 \mu_e + K_e}{\gamma_e (\mu_e + K_e)} A\psi = -2 H(x, y) c(t),
\]
[2.11]
\[
(\lambda - \nu) \psi = \frac{2 K_e}{\gamma_e (\mu_e + K_e)} H(x, y)
\]
[2.12]

where
\[
I = \frac{K_e (2 \mu_e + K_e)}{\gamma_e (\mu_e + K_e)}.
\]

\( \omega = -A\psi \) satisfies the equation
\[
(\lambda - \nu) \omega = \frac{2 \nu}{\gamma_e (\mu_e + K_e)} H(x, y),
\]
\[
\omega = G(x, y) - \frac{2 L(x, y)}{\gamma_e (\mu_e + K_e)} H(x, y),
\]
[2.13]

where \( G(x, y) \) is a solution of \((\lambda - \nu) G = 0\).
\[
A\psi = -G(x, y) + \frac{2}{2 \mu_e + K_e} H(x, y)
\]
and
\[
\psi = G(x, y) + L(x, y),
\]
[2.14]

where \( L(x, y) \) is a solution of Poisson's equation
\[
\Delta \psi = \frac{2}{2 \mu_e + K_e} H(x, y).
\]

Here
\[
\psi = \frac{1}{K_e} [H(x, y) - (\mu_e + K_e) A\psi_0],
\]
[2.15]