Local Ward Identities and the Decay of Correlations in Ferromagnets

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Abstract. Using local Ward identities we prove a number of correlation inequalities for N-component, isotropically coupled, pair interacting ferromagnets; some for all $N \geq 2$ and some for $N = 2, 3, 4$. These are used to prove a mass gap above the mean field temperature, for all $N \geq 2$. For $N = 2, 3, 4$ we prove an upper bound on a critical exponent, and a lower bound on the susceptibility which diverges as $m \to 0$.

1. Introduction

Recently, Dobrushin and Pecherski [1] announced some new results about the possible rates of clustering in equilibrium states of lattice systems with finite range interactions. One of the results is that if the clustering falls-off faster than a certain (dimension dependent) power of the separation, then it is exponential. In the above work, the clustering is expressed by a rather strong condition, which measures the independence of the statistical distribution of spins in any region from all the other spins which are further than a given distance away. Subsequently, in a work published in this issue, Simon [2] formulated, and proved for ferromagnetic Ising models, a new inequality which implies such a property for the two point correlation function. Thus, this inequality leads to an upper bound on the corresponding critical exponent. Furthermore, the inequality was used in [2] to provide upper bounds for the critical temperature of the mass gap. In fact, incorporating an improvement due to Lieb [3], one obtains a sequence of upper bounds, calculable by finite algorithms, which converge to the exact value. The derivation of the mass gap from the above mentioned inequality of [2] is related to its derivation from Griffith's third inequality [4], see [5, 6]. The latter is a particular case of the new inequality, in its improved version of [3].

The main purpose of this note is to prove inequalities similar to those of [2] for multicomponent, ferromagnetic, spin models with $O(N)$ symmetry. We use in the

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derivation local Ward identities [7, 8] and certain correlation inequalities. For the case \( N = 2 \) (plane rotor), a somewhat weaker result, is already contained in [2].

In particular, our results imply a mass gap above the mean field transition temperature for any \( N \geq 2 \), an upper bound on suitable critical exponents for \( N = 2, 3, 4 \), and a lower bound, in terms of \( m \), on the susceptibility, which diverges as \( m \to 0 \).

2. Local Ward Identities

Local Ward Identities were developed by Driessler et al. [7]; similar ideas were discovered by Fröhlich and Spencer [8]. Certain special cases go back to Mermin [9].

We consider a classical system of spins with an a-priori (uncoupled) measure, for which the expectation value is denoted by \( \langle \cdot \rangle_0 \), and an interacting expectation

\[
\langle A \rangle = \langle Ae^{-\beta H} \rangle / \langle e^{-\beta H} \rangle_0.
\]

Let \( \gamma_t \) be a family of automorphisms \([i.e. \gamma_t(AB) = \gamma_t(A)\gamma_t(B)]\) which preserve the a-priori expectation values of the functions, i.e.

\[
\langle \gamma_t(A) \rangle_0 = \langle A \rangle_0,
\]

and let

\[
\dot{A} = \frac{d}{dt} \gamma_t(A)_{t=0}.
\]

Then, the local Ward Identities assert that

\[
\langle \dot{A} \rangle = \beta \langle A \dot{H} \rangle.
\]

This comes from

\[
\frac{d}{dt} \langle \gamma_t(A)e^{-\gamma_t(\beta H)} \rangle_0 = 0
\]

which follows from (2.2).

Driessler et al. applied the local Ward Identities to prove an inequality on the local magnetization function (somewhat similar to Griffith’s third inequality), which was then used to prove the vanishing of the magnetization above, approximately, the mean field transition temperature (the full mean field bound requires for \( N \geq 5 \), an improvement, made in [10]). In the next section we apply a similar procedure to prove an inequality for the correlation function. For temperatures above the mean field transition, the inequality implies a mass gap, by a general argument which, in one way or another, was used in [5, 6, 2] (and which we reformulate in yet another way here).

3. Mass Gap Above the Mean Field Transition Temperature

The systems we consider consist of \( N \)-component spin variables, \( \sigma_i = (\sigma_{i,\lambda}) \) \( \lambda = 1, \ldots, N \), which are associated with lattice sites \( i \in \mathbb{L} = \mathbb{Z}^d \), where \( d \) is the space