Nomenclature

\( B_s \) = acceleration tensors, defined by eq. [2] of (5).
\( c \) = wave velocity of disturbance,
\( D \) = \( \frac{d}{d} \),
\( h \) = half width of channel,
\( I \) = unit matrix,
\( L \) = typical length,
\( P \) = pressure gradient = \(- \frac{\partial p}{\partial x} \),
\( \bar{p} \) = isotropic pressure,
\( R \) = Reynolds number = \( \frac{VL_0}{\mu} \),
\( S \) = \( \frac{g L V}{\bar{p}} \),
\( t \) = time,
\( \bar{\eta}, \bar{\sigma} \) = wave velocity,
\( U_0' \) = \( DU \), evaluated at \( y_0 \),
\( V \) = typical velocity,
\( x, y, z \) = distance variables,
\( \bar{y} \) = \( y/L \),
\( y_0 \) = critical layer, where \( U = c \),
\( \alpha \) = wave number,
\( \varepsilon \) = \([\alpha R U_0']^{-1/3}\),
\( \rho \) = density,
\( \lambda_0 \) = \( \frac{\alpha R U_0'}{S V^{1/2} U_0' + \bar{\sigma}^2} \).
\( \eta \) = stretched coordinate system \( \eta = \frac{y - y_0}{\varepsilon} \),
\( \bar{\mu}, \bar{\omega}_2, \bar{\omega}_3 \) = coefficients of acceleration tensors (material coefficients).
\( \mu, \omega_2, \omega_3, \omega_3 \) = material constants

Abstract

Consideration is given to the stability of plane Poiseuille flow of a slightly viscoelastic fluid which has a constant viscosity and normal stress differences varying nearly with the shear rate. It is shown that the presence of elasticity lowers the critical Reynolds number at which instability occurs.

Introduction

The need for a theoretical study of the stability of flow of viscoelastic fluids has been stressed by Oldroyd (1). This has been underlined by Thomas and Walters (2), (3) and Chan Man Fong and Walters (4). Those authors have found that only a small amount of elasticity can have a marked effect on the stability of the flow.

In this paper we shall carry out a stability analysis for the constitutive equation proposed by White and Metzner (5). The equation can be written as

\[ T = -p I + \bar{\mu} B_1 + \bar{\omega}_2 B_2 + \bar{\omega}_3 B_3, \]  

[1]

where \( \bar{\mu}, \bar{\omega}_2, \bar{\omega}_3 \) are functions of the invariants of the \( B_s \) tensors: \( I_s = tr B_s \), \( II_s = tr B_s^2 \), \( III_s = tr B_s^3 \). By suitable choice of \( \bar{\mu}, \bar{\omega}_2, \bar{\omega}_3 \) eq. [1] can portray a fluid with constant viscosity and with normal stress differences varying asymptotically linearly with the shear rate. This behaviour seems to be in better agreement with the experimental observations of the dilute polymer solutions used in the transition and turbulent flow regime [see, for example, (6) and (7)] than the one employed in (4), where the normal stress differences vary as the square of the shear rate.

Steady State

We consider a plane parallel flow with velocity components, referred to a Cartesian Coordinate System \((x, y, z)\), of the form

\[ v_x = U(y), \quad v_y = 0, \quad v_z = 0. \]  

[2]

The acceleration tensors \( B_s \) can be shown to be

\[
B_1 = \begin{bmatrix} 0 & \frac{dU}{dy} & 0 \\ \frac{dU}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} -2 \frac{dU}{dy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

[3]

On substituting [3] in eq. [1], we obtain

\[ T = \begin{bmatrix} -p + \bar{\omega}_2 \left( \frac{dU}{dy} \right) & -2 \bar{\omega}_2 \left( \frac{dU}{dy} \right)^2 & -p \frac{dU}{dy} \\ \rho \frac{dU}{dy} & -p + \bar{\omega}_2 \left( \frac{dU}{dy} \right)^2 & 0 \\ 0 & 0 & -p \end{bmatrix}. \]

[4]

The normal stress differences are given by

\[ T_{xx} - T_{yy} = -2 \bar{\omega}_2 \left( \frac{dU}{dy} \right)^2, \]

\[ T_{yy} - T_{zz} = \bar{\omega}_2 \left( \frac{dU}{dy} \right)^2. \]

[5]
The second invariant $II_1 = 2 \frac{dU}{dy}^2$.

If we now choose

$$
\bar{\mu} = \mu, \quad \bar{w}_2 = \frac{w_2}{\sqrt{II_1 + w_1^2}}, \quad \bar{w}_3 = \frac{-w_3}{\sqrt{II_1 + w_1^2}},
$$

and it is understood that the positive root is taken in all cases, then the fluid characterized by eqs. [1] and [6] would have a constant viscosity and normal stress differences varying linearly with the shear rate.

The equation of continuity and the stress equations of motion are

$$
\frac{\partial v_i}{\partial x_i} = 0, \quad \epsilon \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \frac{\partial T_{ij}}{\partial x_j}.
$$

On substituting eq. [4] into [8], it can be shown that $U$, subject to the boundary conditions $U = 0$ at $y = \pm h$, is of the form

$$
U = \frac{P}{2\bar{\mu}}(h^2 - y^2).
$$

**Disturbed State**

We now superimpose a small two-dimensional disturbance on the main flow and write

$$
v_x = U(y) + \bar{u}(y) \exp i \alpha (x - ct),
$$

$$
v_y = \bar{v}(y) \exp i \alpha (x - ct),
$$

$$
v_z = 0
$$

The acceleration tensors $B_\alpha$ can be calculated and the dominant term of the second invariant $II$, is $2 (dU/dy)^2$. Hence $w_2$ and $w_3$ have the same form as given by [6]. We can then calculate the stress tensor $T$ and on substituting $T$ into eq. [8], we obtain the analog of the Orr-Sommerfeld equation which can be written in the form

$$
i\alpha (U - c)(D^2 - \alpha^2) \bar{\vartheta} - i\alpha \bar{\vartheta} D^2 U
$$

$$
= \left(1 - \frac{i\alpha (U - c)}{S\sqrt{2}(DU)^2 + w_1^2}\right)(D^2 - \alpha^2)^2 \bar{\vartheta}
$$

$$
+ \frac{i\alpha \bar{v}(DU) + w_3}{S\sqrt{2}(DU)^2 + w_1^2} - \frac{1}{S} D(2(DU)^2 + w_1^2)\bar{\vartheta}^{-1/2}
$$

$$
+ 2i\alpha \bar{v}(U - c) D^2 \bar{\vartheta} - 2i\alpha D^2 \bar{\vartheta} DU
$$

$$
+ D\bar{\vartheta}(- 4i\alpha^2 (U - c) + i\alpha^2 DU - 2i\alpha D^2 U)
$$

$$
- 2i\alpha \bar{v}(\alpha^2 DU + D^2 U))
$$

$$
- \frac{1}{S} D^2 (2(DU)^2 + w_1^2)\bar{\vartheta}^{-1/2}(U - c)(D^2 - \alpha^2) \bar{\vartheta}
$$

$$
+ \bar{\vartheta} D^2 U - 2DU D\bar{\vartheta},
$$

**Solution of the Disturbance Equation**

To solve eq. [11], we employ the usual stretched coordinate technique (see, for example, Lin [8]). We introduce a new coordinate $\eta$ and write

$$
\eta = \frac{y - y_0}{\epsilon}
$$

where $\epsilon = (\alpha R U_0')^{-1/2}$ is a small parameter. We now expand $U, DU, D^2U$ about the critical point $y_0$, in terms of $\epsilon$ and also $\bar{\vartheta}$ as

$$
\bar{\vartheta} = \bar{v}_0 + \epsilon \bar{v}_1 + \cdots
$$

We now substitute the new form of $U, DU, D^2U$ and $\bar{\vartheta}$ into [11] and on retaining the leading terms only, we have

$$
(i\lambda_0 - 1) \frac{d^4 v_0}{d\eta^4} = -i \eta \frac{d^2 v_0}{d\eta^2}.
$$

It can be observed that eq. [14] is similar to eq. [24] of (4). The relationship between $\lambda_0$ in eq. [14] and $\lambda$ in eq. [24] of (4) is

$$
\lambda_0 = \frac{\lambda}{\sqrt{2U_0'' + w_1^2}}.
$$

Thus from the results given in (4) we can conclude that the presence of elasticity destabilizes the flow.

**Conclusion**

The above analysis has been restricted to a two-dimensional disturbance and so $w_2$ does not appear in the analog of Orr-Sommerfeld equation [c. f. (9)]. Thus the stability criterion is independent of the second normal stress difference.

Dugan and Denn (10) and Giesekus (11) have shown that in the Taylor Stability problem the second normal stress difference is of crucial importance.

It would appear desirable to impose a three-dimensional disturbance on the flow and thus be able to investigate the effect of the second normal stress difference. If the 3-dimensional disturbance is more stable than the 2-dimensional disturbance then instability would first occur in the form of 2-dimensional disturbance. Squire (12) has shown that indeed this is the case for the Newtonian fluid and Chang Man Fong (13) has derived a similar result for the viscoelastic fluid designated $B''$ by Beard and Walters (14). On the other hand if the 3-dimensional disturbance is less stable than