the 2-dimensional one then the critical Reynolds number would be even lower than that predicted by the present analysis. Thus the effect of elasticity of the type described by eq. [1] would always have a destabilizing effect on the plane Poiseuille flow.

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References


The Measurement of Normal-Stress Differences Using A Cone-and-Plate Total Thrust Apparatus

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With 4 figures and 1 table

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Abstract

The proposal of Jackson and Kaye (1966) for evaluating both differences of normal stress in a viscometric flow by using only total thrust measurements in a cone-and-plate viscometer is extended. An analytic relation valid for all values of the separation between cone and plate is obtained, which is shown to include as special cases the well-known cone-and-plate formula, the parallel-plate formula of Kotaka et al. (1959) and Jackson and Kaye's.

Examples of the method's application to experimental results are given.

1. Introduction

Recently, Jackson and Kaye (1966) have described a method for obtaining both differences of normal stress from a cone-and-plate rheometer using total thrust measurements only. Their technique involves making measurements of the thrust with the cone displaced axially from the plate, arrangements being otherwise the same as

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for normal rotational testing. The new relation they derive – Jackson and Kaye eq. [17] – involves the evaluation of the derivative of the normal thrust, \( F \), with respect to the distance, \( c \), between the cone apex and plate, in the limit as this distance \( c \) tends to zero. This derivative is obtained by numerical differentiation of experimental point values at the end-point of the range, a notoriously unreliable operation. In the absence of confirmatory parallel-plate observations, values obtained in this way for both differences of normal stress can therefore be accepted only with some reservations.

We shall show below that a more general relation can be obtained relating the derivative of the total normal thrust \( Y \) to the separation distance \( c \), valid for all values of \( c \). This immediately increases the value of the experimental measurements, because it allows values obtained by numerical differentiation from one set of measurements to be tested for consistency. In a statistical sense, a significant degree of uncertainty can be removed.

The generalization is achieved by the use of a more natural coordinate system than that chosen by Jackson and Kaye, and by application of what is often termed ‘the lubrication approximation’ (Pearson 1966, p. 60; 1967). The theory given in the next section shows that both the parallel-plate and the just-touching cone-and-plate are merely special cases of the general axially displaced cone-and-plate situation; that this technique of measuring differences of normal stresses is a result of the particular symmetry of the flow so that the lubrication approximation yields directly two valid terms in an asymptotic expansion in the cone angle \( \beta \); and that the velocity distribution usually chosen for the interpretation of just touching cone-and-plate results (Coleman, Markowitz and Noll, 1965, p. 50; Pearson, 1966, p. 42; Lodge, 1964, p. 202) is not unique to the order of approximation considered.

2. Approximate Theory

The flow region and the cylindrical polar \((r, \theta, z)\) coordinate system naturally suggested by lubrication theory are shown in fig. 1. We assume the plate to be stationary and the cone to rotate about its axis with angular speed \( \Omega \). We assume the fluid free surface to be on the cylinder \( r = R \), noting that this leads to some difficulties in satisfying stress boundary conditions on the interface. This surface is defined solely by the radius of the rim of the cone for geometrical simplicity.

![Fig. 1. The flow region and coordinate system](image)

We now have a problem in fluid mechanics to solve, if we wish to deduce the flow and stress patterns everywhere. The formal way of doing this is to write down the equations of motion, i.e. momentum, mass and energy conservation, to prescribe boundary conditions and to look for a solution, hopefully unique. However, we realize at once that this requires the specification of a constitutive equation for the fluid in question, and this is precisely what we cannot do. Therefore we have to adopt a different approach, as we must in all cases where we cannot prescribe the constitutive equation.

We start by guessing a velocity distribution, naturally making use of any symmetries of the imposed geometry of the boundaries, and satisfying the continuity equation. Next we study the pattern of deformation that results, and decide whether the history of deformation sustained by fluid elements allows any effective simplification of the constitutive equation. If so, we deduce the stress pattern induced rheologically\(^1\)); this will of course include certain general, and perhaps as yet unknown, material functions. Finally we must verify that the velocity and stress pattern so obtained is a valid solution of the equations of motion. If we can show this, we make

\(^1\) For simplicity, we assume here that the energy equation can be decoupled from the other equations of motion. This will not always be the case; when significant temperature variations occur, the procedure is further complicated by having to decide whether to guess or deduce the temperature field.