Rheological models containing fractional derivatives

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With 8 figures in 10 details

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1. Introduction

It has rarely been recognized that fractional derivatives could play an important role in the mathematical formulation of various dynamic phenomena which take place in media with elastic and viscous properties.

As far as we have been able to ascertain, Gemant (1) was the first to describe, in 1936, some oscillatory measurements on an elastic-viscous fluid (carried out by Philippoff) using a generalization of the simple Maxwell model in this sense. Instead of expressing the relationship between shear stress $\tau_{12}$ and shear rate $\dot{\gamma}$ by a Maxwell model,

$$\dot{\gamma} = \frac{1}{G} \frac{d\tau_{12}}{dt} + \frac{\tau_{12}}{\eta}, \quad [1]$$

where $G$ represents the shear modulus and $\eta$ the viscosity, Gemant investigated the following interesting formula:

$$\dot{\gamma} = \frac{1}{(G \cdot \eta)^{1/2}} \frac{d^{1/2}\tau_{12}}{dt^{1/2}} + \frac{\tau_{12}}{\eta}, \quad [2]$$

where $d^{1/2}\tau_{12}/dt^{1/2}$ represents a so-called half differential, which had been mathematically defined about 1830 by the mathematician Abel (2).

The principal reason for the introduction of fractional derivatives by Gemant follows from the investigation of Philippoff (3), since, starting from eq. [1], we can write for the complex dynamic viscosity:

$$\eta(j \omega) = \frac{\tau_{12}(j \omega)}{\dot{\gamma}(j \omega)} = \frac{1 - j \omega \tau}{1 + (\omega \tau)^2}, \quad [3]$$

where $\tau = \eta |G|$, $j = \sqrt{-1}$ and $\omega$ is the angular frequency. Philippoff found the exponent of the argument $(\omega \tau)$ to be not 2, but closer to unity. Eq. [2] does possess this property. Gemant's paper suggests that he did make use of the fractional derivative on the analogy of the problem of the noninductive cable treated by Heaviside calculus.

Scott Blair (4) also used fractional derivatives to describe some dynamic phenomena like creep, stress relaxation, etc., of visco-elastic materials. In his paper he undertakes to describe the behaviour of viscoelastic materials under stress in terms of entities which are not strictly "physical properties". These "quasi-properties" range from entities hardly distinguishable from dimensionally true physical properties to concepts which are much less clearly defined. Quasi-properties measure an ordered process towards equilibrium rather than a state of equilibrium. Scott Blair argues that the Newtonian definition of equality of time intervals, which leads to the concepts of velocity, force, etc., does not apply to quasi-equilibrium states.

In order to maintain the Newtonian time scale, Scott Blair introduces fractional differential equations. The solution of the simplest fractional differential equation relating stress, strain and time is a series equation whose first term is a well-known simple power law (Nutting's equation) to describe the behaviour of many viscoelastic materials under constant stress:

$$\varepsilon = k \cdot \sigma \cdot t^n, \quad [4]$$

where $\varepsilon$ is the strain, $\sigma$ the tensile stress, $t$ the time and in the simple case, both $K$ and $n$ are constants. Scott Blair also pays some attention to the physical meaning of the fractional differential, and he concludes the above mentioned paper by remarking that for the description of the behaviour of the majority of the materials tested by him the fractional differential approach is to be preferred.

The last paper on the use of fractional derivatives in rheology we want to review here, is a general investigation by Slonimsky (5), who has paid some attention to relaxation processes in polymers. Fractional derivatives are proposed by him, because in his opinion the viscoelastic strain is not a result of a simple or complex summation of elasticity and internal friction but an independent type of reversible strain.

In his paper Slonimsky gives an example of the dynamic behaviour of a linear poly-
mer, the viscoelastic strain of which he assumes to be intermediate between the strain of a Hookean elastic body and a Kelvin-Voigt viscoelastic body. Slonimsky shows that solution of the equation of motion for this model leads to expressions fully conforming to the Boltzmann integral equation describing mechanical relaxation processes.

In this contribution we intend to briefly recapitulate the basic mathematics of the fractional derivatives and to treat some aspects of the use of these derivatives for the composition of rheological models. In particular, the behaviour of the fractional derivative model will be examined in various conventional experiments on textile fibres. Furthermore, some experimental data will be analysed in terms of the proposed model containing fractional derivatives. Finally we will pay some attention to the merits of this descriptive method and sketch the relationship between the model and the more familiar expressions of the theory of linear viscoelasticity.

2. Mathematical formulation

There are several ways to establish a generalized differential and integral calculus of any fractional order. Here we will exemplify a simple process of generalization of the concepts of differentiation and integration as indicated by Courant (2) and Gemant (6).

First we want to refresh the basic knowledge about normal integral and differential calculus relating to our subject.

If the function $F_1(x)$ is differentiable for $x > 0$, $F_1(0_+) = 0$, and the function $F_2(x)$ is continuous, then it can be proved that for $x > 0$:

$$\frac{d}{dx} \int_0^x F_1(x - \xi) \cdot F_2(\xi) \cdot d\xi = \int_0^x F_1^{(1)}(x - \xi) \cdot F_2(\xi) \cdot d\xi + F_1(0_+) \cdot F_2(x).$$  \[5\]

Resuming, both functions $F_1(x)$ and $F_2(x)$ must meet the conditions:

a) that they are defined for $x > 0$,
b) that they are integrable in any finite interval $[a, b]$, where $0 < a < b < \infty$, and
c) that $\lim_{\xi \to 0} \int_{-\infty}^\xi |F(x)| \cdot dx < \infty$, where $d$ is an arbitrary positive number.

To find a general rule for the integration procedure, eq. [5] is differentiated $(n-1)$ times and the function $F_1(x)$ is chosen in such a manner that the function itself and its first $(n-1)$ derivatives vanish at $x=0$. Eq. [5] then changes into

$$\frac{d^n}{dx^n} \int_0^x F_1(x - \xi) \cdot F_2(\xi) \cdot d\xi = \int_0^x F_1^{(n)}(x - \xi) \cdot F_2(\xi) \cdot d\xi.$$

To meet the condition specified above we choose the function $F_1(x - \xi)$ to be equal to $(x - \xi)^n$, so eq. [6] becomes

$$\int_0^x F_2(\xi) \cdot d\xi = \frac{d^n}{dx^n} \int_0^x (x - \xi)^n \cdot F_2(\xi) \cdot d\xi.$$

If we put $f(x) = \frac{d}{dx} \int_0^x F_2(\xi) \cdot d\xi$ and take the $(n+1)$-fold integral of the function $f(x)$ between the limits 0 and $x$, then

$$F(x) = \int_0^x \cdots \int_0^x f(x) \cdot dx = \int_0^x \frac{(x - \xi)^n}{n!} \cdot F_2(\xi) \cdot d\xi.$$  \[8\]

Eq. [8] expresses the useful result that a $(n+1)$-fold integral of the function $f(x)$ can be replaced by a single integration of the function

$$\frac{(x - \xi)^n}{n!} \cdot F_2(\xi)$$

to $\xi$, where $f(x) = \frac{d}{dx} \int_0^x F_2(\xi) \cdot d\xi$.

In eq. [8] we have restricted ourselves by allowing $n$ only to be an integer. However, by the introduction of Legendre's gamma function,

$$\Gamma(z) = \int_0^\infty e^{-t} \cdot t^{z-1} \cdot dt,$$  \[9\]

eq [8] does acquire a meaning for any value of the argument $z$. So eq. [8] is a possible definition of generalized integrals. In ref. (7) one may find the mathematical proof that the definition eq. [9] holds for any value of $z$, except the points $z = 0, -1, -2, \ldots$ etc. The gamma function satisfies the functional equation $\Gamma(z + 1) = z \cdot \Gamma(z)$; hence, if $z$ is a positive integer $n$, $\Gamma(n) = (n-1)!$

Accordingly, eq. [8] changes into

$$F(x) = \frac{1}{\Gamma(n+1)} \int_0^x (x - \xi)^n \cdot F_2(\xi) \cdot d\xi.$$  \[10\]

If the symbol $D$ denotes the differential operator and again, if $D^{-1}$ denotes the integral