A PERTURBATION BOUND FOR THE GENERALIZED POLAR DECOMPOSITION

REN-CANG LI

Department of Mathematics, University of California at Berkeley, Berkeley, California 94720, U.S.A.

Abstract.

Let A be an m x n complex matrix. A decomposition $A = QH$ is termed a generalized polar decomposition of A if $Q$ is an m x n subunitary matrix (sometimes also called a partial isometry) and H a positive semidefinite Hermitian matrix. It was proved that a nonzero matrix $A \in \mathbb{C}^{m \times n}$ has a unique generalized polar decomposition $A = QH$ with the property $\mathcal{R}(Q^H) = \mathcal{R}(H)$, where $Q^H$ denotes the conjugate transpose of $Q$ and $\mathcal{R}(H)$ the column space of $H$. The main result of this note is a perturbation bound for $Q$ when $A$ is perturbed.

AMS Subject Classifications: 15A45.

Keywords: Generalized polar decomposition, Perturbation bound.

Throughout the paper, we use the following notation. $\mathbb{C}^{m \times n}$ is the set of m by n complex matrices; $\mathbb{C}_r^{m \times n} \subset \mathbb{C}^{m \times n}$ is the set of m x n complex matrices having rank r; and $\mathbb{U}_n \subset \mathbb{C}^{n \times n}$ is the set of n x n unitary matrices. For $A \in \mathbb{C}^{m \times n}$, $\mathcal{R}(A)$ denotes the column space of A. Further $A^H$ and $A^+$ denote the conjugate transpose and Moore-Penrose inverse of $A$, respectively. $I^{(n)}$ is the n x n unit matrix and $I^{(r)}_{m,n} \in \mathbb{C}_r^{m \times n}$ is defined by

$$I^{(r)}_{m,n} \equiv \begin{pmatrix} I^{(r)} & 0 \\ 0 & 0 \end{pmatrix}.$$  

$\| \cdot \|_2$ is used to denote the Euclidean length of a vector or the spectral norm of a matrix and $\| \cdot \|_F$ the Frobenius norm of a matrix.

First of all, let us summarize some essential definitions and properties concerning a generalized polar decomposition (for details, see Sun & Chen [6]).

$Q \in \mathbb{C}^{m \times n}$ is an m x n unitary matrix if $Q^H Q = I^{(n)}$ ($m \geq n$) or $Q Q^H = I^{(m)}$ ($m \leq n$); and $Q \in \mathbb{C}^{m \times n}$ an m x n subunitary matrix (sometimes also called an m x n partial isometry) if $\|Qx\|_2 = \|x\|_2$ for all $x \in \mathcal{R}(Q^H)$. Every matrix $A \times \mathbb{C}^{m \times n}$ can be decomposed as $A = QH$, where $Q \in \mathbb{C}^{m \times n}$ is unitary or subunitary and $H \in \mathbb{C}^{n \times n}$ Hermitian positive semidefinite. The decomposition such that $Q$ is unitary is called a polar decomposition.

decomposition of $A$, otherwise a generalized polar decomposition. With the help of singular value decomposition (SVD), we can construct a few generalized polar decompositions of a matrix. Let

$$A = U \Sigma V^H, \quad \Sigma = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}$$

be a SVD of $A \in \mathbb{C}^{m \times n}$, where $r \leq \min\{m, n\}$, $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$, $\Omega = \text{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i > 0$, $i = 1, \ldots, r$, then for any integer $p \geq r$, we have

$$A = U \Sigma V^H = (U I_{m,n}^{(p)} V^H) \left[ V \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} V^H \right] = Q_p H,$$

where

$$Q_p \equiv U I_{m,n}^{(p)} V^H \quad \text{and} \quad H \equiv V \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} V^H,$$

$p$ running from $r$ to $\min\{m, n\}$. It was proved that the decomposition $A = QH$ is unique under the condition:

$$R(Q^H) = R(H),$$

where $Q \in \mathbb{C}^{m \times n}$ is subunitary and $H \in \mathbb{C}^{n \times n}$ positive semidefinite Hermitian (see, e.g., Ben-Israel & Greville [1, p. 255]). Thus the unique decomposition $A = QH$ satisfying (4) can be given by (2) and (3) with $p = r$.

Perturbation bounds associated with the polar decomposition of a nonsingular matrix can be found in [2, 3, 4, 5]. In the following we will present a perturbation bound for the subunitary factor $Q$ in the generalized polar decomposition.

**LEMMA 1.** Let $U \in \mathbb{C}^n$ and

$$\Sigma = \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{r \times r}, \quad \tilde{\Sigma} = \begin{pmatrix} \tilde{\Omega} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{r \times r},$$

$$\Omega = \text{diag}(\sigma_1, \ldots, \sigma_r), \quad \tilde{\Omega} = \text{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r).$$

Then

$$\|U \Sigma - \tilde{\Sigma} V\|_F \geq \min_{1 \leq i, j \leq r} \{\sigma_i, \tilde{\sigma}_j\} \|U I_{m,n}^{(r)} - I_{m,n}^{(r)} V\|_F.$$