HIGH-ORDER ADAPTIVE FINITE ELEMENT-SINGLY IMPLICIT RUNGE-KUTTA METHODS FOR PARABOLIC DIFFERENTIAL EQUATIONS

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Abstract.

We describe an adaptive mesh refinement finite element method-of-lines procedure for solving one-dimensional parabolic partial differential equations. Solutions are calculated using Galerkin’s method with a piecewise hierarchical polynomial basis in space and singly implicit Runge-Kutta (SIRK) methods in time. A modified SIRK formulation eliminates a linear systems solution that is required by the traditional SIRK formulation and leads to a new reduced-order interpolation formula. Stability and temporal error estimation techniques allow acceptance of approximate solutions at intermediate stages, yielding increased efficiency when solving partial differential equations. A priori energy estimates of the local discretization error are obtained for a nonlinear scalar problem. A posteriori estimates of local spatial discretization errors, obtained by order variation, are used with the a priori error estimates to control the adaptive mesh refinement strategy. Computational results suggest convergence of the a posteriori error estimate to the exact discretization error and verify the utility of the adaptive technique.

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Key words: singly implicit Runge-Kutta methods, finite-element methods, adaptive mesh refinement, error estimation.

1. Introduction.

Basic adaptive strategies for the solution of parabolic partial differential equations (PDEs) include method-of-lines (MOL) approaches with spatial mesh (h-refinement) [1, 3, 7], mesh motion (r-refinement) [1, 4, 5, 15], and order variation (p-refinement) [12]. Solution “enrichment indicators,” which are frequently esti-
mates of local spatial discretization errors [1, 3, 7], are obtained from preliminary solutions and used to identify portions of the domain in need of additional resolution. Some combination of the three basic adaptive enrichment strategies [1, 4] is used to alter the discretization and recursively calculate improved solutions until specified accuracy criteria have been satisfied.

With aims of (i) examining high-order methods and hp-refinement, (ii) exploring optimal spatial and temporal refinement strategies, and (iii) studying local refinement methods that simultaneously address adaptive temporal and spatial enrichment [12], we consider an h-refinement procedure for the solution of $M$-dimensional vector parabolic problems of the form

\begin{align}
  u_t + f(x, t, u, u_x) &= D(x, t, u, u_x), \quad x \in \Omega = (c, d), \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad c = \Omega, \\
  u_i(x, t) &= c_i(t), \quad x \in \partial \Omega_i^E, \\
  D_i(x, t, u, u_x) &= c_i(t), \quad x \in \partial \Omega_i^N, \quad i = 1, 2, \ldots, M, \quad t > 0.
\end{align}

The boundary $\partial \Omega = \partial \Omega_i^E + \partial \Omega_i^N$ is divided component-wise, $i = 1, 2, \ldots, N$, into sets where essential (E) and natural (N) boundary data is applied.

We solve (1.1) by a finite element Galerkin MOL with a piecewise polynomial hierarchical spatial basis of degree $p \geq 1$ (§2) and a singly implicit Runge-Kutta (SIRK) method in time (§2). SIRKs are well-suited to adaptive computations which require frequent restarts of the temporal integration due to spatial enrichment, a consideration that will be even more critical when local refinement methods are addressed [19]. We consider SIRKS where all stages have the same order of accuracy as the solution [10]. This factor may be important in eliminating order reduction [11, 17] when some Runge-Kutta methods are used to solve PDEs. Stability considerations and temporal error estimates of each stage [19] permit the acceptance of solutions at some intermediate stages whenever the final solution lacks the necessary accuracy. This aspect of the SIRK method greatly improves efficiency since very few integration steps will require total rejection.

In §2, the spatial and temporal discretization schemes are presented. A reduced-order interpolating polynomial, described in §2.1, appears to produce better initial guesses for Newton iteration than the standard SIRK interpolation formulas. In §2.2, we present stability results and temporal error estimates using a single additional stage for solutions at intermediate stages. Local (in time) a priori energy error estimates are established for nonlinear scalar problems in §3 and an adaptive h-refinement algorithm is described in §4. A posteriori error estimates are obtained by using the hierarchical basis to create an order embedding and nodal superconvergence to simplify the complexity. Local estimates of the spatial errors are used as refinement indicators to identify regions of the domain that require more or less resolution. Two examples, presented in §5, verify the accuracy of the a posteriori error estimation procedure and the utility of the local refinement algorithm. A brief