Secondary flow in a parallel-disk viscometer

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With 6 figures

(Received June 1, 1970 [Received copy accepted December 2, 1970])

Introduction

The parallel-disk, or parallel-plate, viscometer, in which a fluid is sheared between a rotating and a stationary circular disk, is usually analysed by assuming that inertial effects are absent. In that case fluid streamlines are circular and the torque is directly proportional to the fluid viscosity. Significant inertial effects in a parallel-disk viscometer have been reported recently by Savins and Metzner (1), who found that dye penetration experiments to study the secondary flow could be described in part by an approximate solution of the Navier–Stokes equations for infinite parallel disks. Mellor et al. (2) have also found at Reynolds Numbers an order of magnitude higher than those used in viscometers that the infinite disk solution adequately represents the flow between rotating finite disks.

Savins and Metzner have further noted that the secondary flow appears to consist of numerous concentric cells with a radial spacing of the order of the spacing between the disks. Such a cellular pattern is incompatible with the infinite disk solution used to describe their penetration data and appears to be in conflict with other flow visualization experiments (3, 4). This work presents the results of a numerical solution of the Navier–Stokes equations for flow between finite disks with a free outer surface. The computed inertial secondary flow consists of a single eddy which, for large radius-to-gap ratios, is closely represented by the solution for infinite disks.

Basic equations

The flow geometry is shown in fig. 1. In a cylindrical ($\rho$, $\theta$, $z$) coordinate system the rotating disk is taken to be at $z = 0$ and the stationary disk at $z = H$. The free liquid–air interface, which for computational simplification is assumed to be planar, is at $\rho = R$.

The lower disk rotates with angular velocity $\Omega$. Assuming angular symmetry, the steady state Navier–Stokes and continuity equation for an incompressible Newtonian fluid are

$$\begin{align*}
\frac{\partial \bar{u}}{\partial \rho} - \frac{\partial \bar{v}}{\partial z} + \bar{w} \frac{\partial \bar{u}}{\partial z} &= - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial \rho} \\
+ \bar{v} \left[ \frac{\partial ^2 \bar{u}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \bar{u}}{\partial \rho} - \frac{\bar{u}}{\rho^2} + \frac{\partial ^2 \bar{u}}{\partial z^2} \right] \\
\frac{\partial \bar{v}}{\partial \rho} + \frac{\bar{v}}{\rho} + \bar{w} \frac{\partial \bar{v}}{\partial z} &= \rho \left[ \frac{\partial ^2 \bar{v}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \bar{v}}{\partial \rho} - \frac{\bar{v}}{\rho^2} + \frac{\partial ^2 \bar{v}}{\partial z^2} \right] \\
\frac{\partial \bar{w}}{\partial \rho} + \frac{\bar{v}}{\rho} + \frac{\partial \bar{w}}{\partial z} &= 0
\end{align*}$$

Here, $\bar{u}$, $\bar{v}$, and $\bar{w}$ are, respectively, the radial, angular, and axial components of velocity. $\bar{p}$ is the pressure, $\bar{q}$ the density, and $\nu$ the kinematic viscosity. The boundary conditions resulting from no-slip at the solid surfaces, symmetry about the axis of rotation, and a free interface at the outer radius are as follows:

$$\begin{align*}
\rho = 0 & : \bar{u} = \bar{v} = 0; \bar{w} = \bar{r} \Omega \\
\rho = H & : \bar{u} = \bar{v} = \bar{w} = 0 \\
\rho = 0 & : \bar{u} = \bar{v} = \partial \bar{w}/\partial \rho = 0 \\
\rho = R & : \bar{u} = \frac{\partial \bar{w}}{\partial \rho} = 0; \frac{\bar{v}}{\rho} + \frac{\partial \bar{w}}{\partial z} = 0
\end{align*}$$

It is convenient to define dimensionless variables as follows:

$$\begin{align*}
u = \bar{u}/R \Omega & \quad v = \bar{v}/R \Omega & \quad w = \bar{w}/R \Omega \\
r = \bar{r}/R & \quad z = \bar{z}/H & \quad p = \bar{p}/R^2 \Omega^2.
\end{align*}$$

Eq. [1] through [8] are then rewritten...
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\[ u \frac{\partial u}{\partial r} - \frac{v^2}{r} + \Lambda w \frac{\partial u}{\partial z} = - \frac{\partial p}{\partial r} + \frac{1}{N_{Re}} \left( \frac{1}{A^2} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - u \right) + \frac{\partial^2 u}{\partial z^2} \right) \]  

\[ \frac{\partial v}{\partial r} + \frac{u v}{r} + \Lambda w \frac{\partial v}{\partial z} \]

\[ = \frac{1}{N_{Re}} \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r} \right) + \frac{\partial^2 v}{\partial z^2} \]  

\[ \frac{\partial w}{\partial r} + \frac{A w}{\partial z} = - A \frac{\partial p}{\partial r} + \frac{1}{N_{Re}} \left( \frac{1}{A^2} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2} \right) \]

\[ u + \frac{\partial u}{\partial r} + \Lambda w \frac{\partial u}{\partial z} = 0 \]

at \( z = 0 \):

\[ u = w = 0; \quad v = r \]

\[ z = 1 \]:

\[ u = v = w = 0 \]

\[ r = 0 \]:

\[ u = v = w \frac{\partial w}{\partial r} = 0 \]

\[ r = 1 \]:

\[ u = \frac{\partial w}{\partial r} = 0; \quad \frac{\partial v}{\partial r} - \frac{v}{r} = 0 \]

The problem now depends upon only two parameters,

\[ A = \frac{R}{H} \quad N_{Re} = \frac{H^2 \Omega}{v} \].

The pseudo-Reynolds Number, \( N_{Re} \), is used to conform with previous studies (1, 2). The dimensionless quantity which has the meaning of the ratio of inertial to viscous forces is the product \( A N_{Re} \) but this group does not appear in infinite disk analyses.

The inertial secondary flow is shown graphically as a projection of the velocity vector in the \( r-z \) plane for the four cases in figs. 2 to 5, respectively. As can be seen, the numerical solution indicates a single eddy centered near the free surface. The flow for \( A = 10 \) was perturbed in \( w \) in the region.

\[ u = - \frac{r N_{Re}}{60} \left[ 4(1-z) - 9(1-z)^2 + 5(1-z)^3 \right] \]

\[ v = r(1-z) - \frac{r N_{Re}^2}{6300} \left[ 5 - 8(1-z) - 35(1-z)^2 \right] \]

\[ + 63(1-z)^3 - 20(1-z)^4 \]

\[ w = \frac{N_{Re}}{60 A} \left[ - 4(1-z)^3 + 6(1-z)^2 - 2(1-z)^2 \right] \]

\[ p = \frac{3}{20} r^2 + \frac{1}{30 A^2} \left[ - 4(1-z)^3 \right] + \frac{9}{2} (1-z)^4 + \frac{5}{2} (1-z)^3 \]

Eqs. [18] through [21], taken to first order in \( N_{Re} \), are equivalent to the solution obtained by Savins and Metzner (1). For \( N_{Re} \) of order unity the second order correction term in eq. [19] is always negligible.

**Numerical solution**

Eqs. [9] through [16] were solved numerically using a relaxation method of Chorin (5). The method simulates the time behavior of a compressible system which has a steady state described by the steady state incompressible Navier-Stokes and continuity equations. Finite differencing is based on the explicit Dufort-Frankel scheme. Details of the method and modifications required for this problem are described elsewhere (6). Twenty grid spaces in both \( r \) and \( z \) directions were used.

Solutions were obtained for values of the radius-to-gap ratio \( A = 1, 2, 5, \) and \( 10 \) for \( N_{Re} = 1 \). This Reynolds Number is within the range of usual interest in viscometry. In all cases the maximum deviation of the angular component of velocity from the inertialess solution, eq. [17], was no more than one percent. The inertial secondary flow is shown graphically as a projection of the velocity vector in the \( r-z \) plane for the four cases in figs. 2 to 5, respectively. As can be seen, the numerical solution indicates a single eddy centered near the free surface. The flow for \( A = 10 \) was perturbed in \( w \) in the region.