ON A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH PERIODIC IMPULSE COEFFICIENTS*

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Abstract

A thorough investigation of the system
\[ \frac{d^2y(x)}{dx^2} + p(x)y(x) = 0 \]
with periodic impulse coefficients
\[ p(x) = \begin{cases} 1, & 0 < x < x_0(2\pi > x_0 > 0) \\ -\pi, & x_0 < x < 2\pi(n > 0) \\ p(x) = p(x+2\pi), & -\infty < x < \infty \end{cases} \]
is given, and the method can be applied to one with other periodic impulse coefficients.

I. The Problem and Results

The system of second order differential equations
\[ \frac{d^2y(x)}{dx^2} + p(x)y(x) = 0 \]  
with periodic coefficient \( p(x) \) has wide applications and attracts much attention. People interested in it may find literatures in [1].

By taking a new variable
\[ u(x) = \left( \frac{dy(x)}{dx} \right)/y(x), \]
the system (1.1) is transformed into the Riccati Equation
\[ \frac{du(x)}{dx} = u^2(x) + p(x) \]
with periodic coefficient \( p(x) \). Since, in general, the Riccati Equation possesses nonexplicit elementary solution, the progress made in this respect is rather slow, and even for some very common cases, thorough investigation is still lacking. For example, recently the case

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is tackled by Pu [2], and some sufficient conditions for oscillatory and nonoscillatory properties are first given.

For this type of step functions \( p(x) \), this paper gives the general method, applied to the above problem as a concrete example to obtain the following results:

(a) The explicit exact general analytic solutions in Section II.
(b) The necessary and sufficient conditions of the existence of periodic solutions and their explicit forms in Section III.
(c) The necessary and sufficient conditions of the stability properties of the trivial solutions in Section IV.
(d) The necessary and sufficient conditions of the oscillatory and nonoscillatory properties of solutions, and of the existence of almost periodic solutions.
(e) The explicit general solutions of the corresponding Riccati Equation, and the necessary and sufficient conditions of the existence of periodic continuous solutions in concrete forms.

II. Explicit Exact General Analytic Solutions

We denote system (1.1) under conditions (1.4) by \( D(x_0, \eta) \). The first problem is to obtain explicit general solutions of \( D(x_0, \eta) \).

First of all, two systems \( D(0, \eta) \) and \( D(2\pi, \eta) \) are trivial. The former is

\[
D(0, \eta) : -\eta y''(x) + \eta y(x) = 0, \quad 0 < x < 2\pi, \quad \eta > 0
\]

with general solutions:

\[
y(x) = c_1 \exp(\sqrt{\eta} x) + c_2 \exp(-\sqrt{\eta} x);
\]  

and the latter is

\[
D(2\pi, \eta) : \frac{d^2 y(x)}{dx^2} + y(x) = 0, \quad 0 < x < 2\pi,
\]

with general solutions:

\[
y(x) = c_1 \cos x + c_2 \sin x;
\]  

where \( c_1 \) and \( c_2 \) are two arbitrary constants.

In the following we shall investigate the general case:

\[0 < x_0 < 2\pi, \quad \eta > 0.\]  

In [2], a method of calculation of successive intervals is used, and the properties of series are discussed to obtain some sufficient conditions for oscillatory properties. In this paper the global solutions in one explicit general form are given.

According to the well-known classical theory of Floquet [3] and of Zamysh [3], the general solutions of the system \( D(x_0, \eta) \) can be written in the following form:

\[
y(x) = c_1 \exp(\sigma x_0) f_1(x) + c_2 \exp(\sigma x_0) f_2(x),
\]

\[
\sigma = \pm \sqrt{\eta} \quad \text{or} \quad \sigma = \pm \sqrt{-\eta}.
\]