NECESSARY AND SUFFICIENT CONDITIONS
FOR THE EXISTENCE OF PERIODIC SOLUTIONS
OF RICCATI’S EQUATION WITH PERIODIC
COEFFICIENTS*

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Abstract

In this paper, the concept of generalized \( \omega \)-periodic solution is given for Riccati’s equation
\[ y' = a(t)y^2 + b(t)y + c(t) \]
with periodic coefficients, the relation between generalized \( \omega \)-periodic solutions and the characteristic numbers of system
\[ x' = c(t)x, \quad x' = -a(t)x - b(t)x, \]
is indicated, and several necessary and sufficient conditions are given using the coefficients. Moreover, in the case of \( a(t) \)
without zero, the relation between the number of continuous \( \omega \)-periodic solutions of \( y' = a(t)y^2 + b(t)y + c(t) + \delta \)
and the parameter \( \delta \) is given; thus the problem on the existence of continuous \( \omega \)-periodic solutions is basically solved.

§ 1. Origination of the Problem

We consider Riccati’s equation
\[ y' = a(t)y^2 + b(t)y + c(t) \quad (1) \]
where \( a(t) \), \( b(t) \) and \( c(t) \) are all continuous \( \omega \)-periodic functions (\( \omega > 0 \)). If all \( a(t) \), \( b(t) \) and \( c(t) \) are constant functions, or either of \( a(t) = 0 \) and that \( c(t) = 0 \) holds, the properties of solutions of (1) are well known. So we suppose that \( a(t) \neq 0 \), \( c(t) \neq 0 \) and \( \omega \) is the smallest period of one (at least) of \( a(t) \), \( b(t) \) and \( c(t) \). During his lecturing in China in 1978, Chen Xingshen posed a problem on the existence of continuous \( \omega \)-periodic solutions (\( \omega \)-CPSs) of (1). Qin Yuanxun immediately gave the solution [1] or [2] to the problem in the case of \( a(t) \) without zero. He indicates that if the polynomial of \( y \) on the right side of (1) has two continuous real roots, \( y_1(t) \) and \( y_2(t) \), with \( \max y_1(t) < \min y_2(t) \), then (1) has two \( \omega \)-CPSs, one of which is asymptotically stable and the other unstable; and that (1) has no \( \omega \)-CPS if \( b^2(t) - 4a(t)c(t) < 0 \).

Does (1) have any \( \omega \)-CPS if both conditions mentioned above are not satisfied? How about the number of \( \omega \)-CPSs? How to make a judgement? How about the conclusion in the general case (i.e. \( a(t) \) may have zeros)? We will answer these questions in this paper.

§ 2. Generalized ω–Periodic Solution and Characteristic Number

We consider the differential system
\[ \dot{x} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 & c(t) \\ -a(t) & -b(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A(t)x. \tag{2} \]

Let \( y(t, y_0, t_0) \) \((t \in (t_1, t_2))\) be the solution of (1) with \( y(t_0, y_0, t_0) = y_0 \) and \( x(t, x_0, t_0) = (x_1(t, x_0, t_0), x_2(t, x_0, t_0)) \) be the solution of system (2) with \( x(t_0, x_0, t_0) = x_0 \).

Then \( y(t, y_0, t_0) = \frac{x_1(t, x_0, t_0)}{x_2(t, x_0, t_0)} \) \((t \in (t_1, t_2))\) if \( x_0 = (y_0, 1)^T \).

A. Generalized Solution

Consider the following equation
\[ y' = -c(t)y^2 - b(t)y - a(t). \tag{1}' \]

For the sake of convenience, we first generalize the concept of continuous inexpan-sible solutions.

It is known that any open set in \( R \) is composed of mutually nonintersecting open intervals, which are called fundamental open intervals. An open set has only one fundamental open interval if the set itself is an open interval.

**Definition 1.** A function \( y(t) \) which is defined on an open set \( J \subset R \) is called a generalized solution of (1) if it satisfies the following conditions: (i) On each of the fundamental open intervals of \( J \), \( y(t) \) is a continuous inexpansible solution of (1); (ii) Let \( I = \{t: t \in J \text{ and } y(t) = 0\} \) and
\[ Y(t) = \begin{cases} 0, & t \in (R-J) \text{ if } R-J \text{ is not empty;} \\ \frac{1}{y(t)}, & t \in (J-I). \end{cases} \]

Then on each of the fundamental open intervals of \( R-I \), \( Y(t) \) is a continuous inexpansible solution of (1)' if \( y(t) \) also satisfies \( y(t_0) = y_0 \), we rewrite it as \( y(t, y_0, t_0) \). If there is a unique function \( y(t) \) satisfying (i), (ii) and \( y(t_0) = y_0 \), we call \( y(t, y_0, t_0) \) a unique generalized solution of (1) for \( (t_0, y_0) \).

Similarly, we can define a generalized solution of (1)'.

It is apparent that a generalized solution \( y(t) \) of (1) is a continuous solution on \( R \) in the usual sense if \( J = R \).

**Proposition 1.** Suppose \( y_0 \neq 0 \). If \( y(t, y_0, t_0) \) defined on an open set \( J \subset R \) is a generalized solution of (1), then
\[ Y(t, \frac{1}{y_0}, t_0) = \begin{cases} 0, & t \in (R-J) \text{ if } R-J \text{ is not empty;} \\ \frac{1}{y(t, y_0, t_0)}, & t \in (J-I) \end{cases} \]
is a generalized solution of (1)'.

The proof is obvious.

**Proposition 2.** For any \( (t_0, y_0) \), there is a unique generalized solution \( y(t, y_0, t_0) \) of (1). Moreover, \( y(t, y_0, t_0) = \frac{x_1(t, x_0, t_0)}{x_2(t, x_0, t_0)} \), where \( x_0 = (y_0, 1)^T \) and \( x = (x_1, x_2)^T \).