BOUNDS OF EIGENVALUES OF A GRAPH

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Abstract

Let $G$ be a simple graph with $n$ vertices. We denote by $\lambda_i(G)$ the $i$-th largest eigenvalue of $G$. In this paper, several results are presented concerning bounds on the eigenvalues of $G$. In particular, it is shown that $-1 \leq \lambda_2(G) \leq (n-2)/2$, and the left hand equality holds if and only if $G$ is a complete graph with at least two vertices; the right hand equality holds if and only if $n$ is even and $G \cong 2K_{n/2}$.

§ 1. Introduction

Let $G$ be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Then its adjacent matrix $A(G)$ is defined to be an $n \times n$ matrix $(a_{ij})$ where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. It follows immediately that if $G$ is simple graph, then $A(G)$ is a symmetric $(0, 1)$-matrix in which every diagonal entry is zero. We shall denote the characteristic polynomial of $A(G)$ by $P(G, x)$, and refer to it as the characteristic polynomial of $G$. Since it is uniquely determined by the graph $G$. Thus

$$P(G, x) = \det(xI - A) = \sum_{i=0}^{n} a_{ii} x^{n-i}.$$  

Since $A(G)$ is a real symmetric matrix, its eigenvalues (the roots of the polynomials) must be real and may be ordered as

$$\lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \cdots \geq \lambda_n(A(G)).$$

These eigenvalues are called eigenvalues of $G$ and the sequence of $n$ eigenvalues is called the spectrum of $G$. Denote $\lambda_i(A(G))$ by $\lambda_i(G)$. For a few special classes of graphs, including complete graph, cycles, paths and complete bipartite graphs, exact values for the spectrum are known\cite{5,7,10}. Since eigenvalues are often difficult to evaluate, it is sometimes useful to obtain bounds for them. Several bounds have been found for the spectral radius of certain graph classes\cite{11,4,5,8,10}. Recently some bounds have appeared for the remaining eigenvalues of the spectrum\cite{2,5,9}. In this paper, we obtain some bounds for eigenvalues of graphs.

§ 2. Lemmas

Lemma 1. (the Courant–Weyl inequality) Let $\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)$ ($\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$) be the eigenvalues of a real symmetric matrix $X$. If $A$ and $B$ are real symmetric matrices of order $n$, and if $C = A + B$, then

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\[\lambda_{n-k-1}(B) > \lambda_{n-k}(A) + \lambda_{n-1}(B),\]
\[\lambda_{n+1}(B) < \lambda_{n+1}(A) + \lambda_{n+1}(B),\]
where \(0 \leq k, 0 \leq l, 0 \leq s, 0 \leq t, k + l + 1 \leq n\) and \(s + t + 1 \leq n\).

**Proof.** See [6].

**Lemma 2.** Let \(G\) be a simple graph with \(n \geq 2\) vertices, and \(G^c\) be the complement of \(G\). Then
\[\lambda_k(G) + \lambda_{n-k}(G^c) \leq -1 < \lambda_k(G) + \lambda_{n-k}(G^c) \quad (\delta \geq 2).\]

**Proof.** This follows from Lemma 1 by taking \(A = A(G), B = A(G^c), C = A(K_n),\)
\(\delta = n - k, l = \delta - 2, t = \delta - 1\) and \(s = n - \delta\).

It is easy to obtain the following

**Lemma 3.** Let \(G\) be a complete bipartite graph with \(n\) vertices. Then
\[\lambda_n(G) \geq - \sqrt{\left\lceil \frac{n}{2} \right\rceil \left(\frac{n+1}{2}\right)} \geq - n/2,\]
where \([x]\) denotes the largest integer not greater than \(x\).

Let \(V'\) be a subset of vertices of a graph \(G\) and \(|V'| = k\). Denote by \(G - V'\) the subgraph obtained from \(G\) by deleting the vertices in \(V'\) together with incident edges. If \(V' = \{v\}\), we write \(G - v\) for \(G - \{v\}\).

**Lemma 4.**
\[\lambda_n(G) \geq \lambda_n(G - V') \geq \lambda_{n-k}(G) \quad (1 \leq k \leq n - k).\]

**Proof.** See [5] or [10].

**Lemma 5.** Let \(G\) be a simple connected graph with \(n\) vertices. If \(n \geq 2\) and \(G\) is not complete graph, then
\[\lambda_n(G) \geq 0.\]

**Proof.** Since \(G\) is not a complete graph, there exist two vertices \(u\) and \(v\) such that \(u\) is not adjacent to \(v\). Taking \(V' = V(G) - \{u, v\}\), by Lemma 4, we have
\[\lambda_n(G) \geq \lambda_n(G - V') = 0.\]

**Lemma 6.** Let \(G\) be a simple graph with \(n\) vertices. If \(G\) have \(n - 1\) negative eigenvalues, then \(G\) is isomorphic to \(K_n\).

**Proof.** By Lemma 5, we have \(\lambda_n(G) = -1\) and \(G\) is isomorphic to \(K_n\).

§ 3. Main Results

The following Theorem is due to \(G\). Constantine [3], now we give a simpler proof.

**Theorem 1.** Let \(G\) be a simple graph with \(n\) vertices. Then
\[\lambda_n(G) \geq \lambda_n(K_{(n^2, (n+1)/2)}) = - \sqrt{\left\lceil \frac{n}{2} \right\rceil \left(\frac{n+1}{2}\right)} \geq - n/2,\]
where we denote by \([x]\) the largest integer not greater than \(x\).

**Proof.** Since \(\lambda_n(G) \geq 0\), we assume \(n \geq 2\). Let \(X_1\) be a unit eigenvector of \(A(G)\) associated with \(\lambda_n(G) (1 \leq i \leq n)\) and let \(X = (x_1, x_2, \ldots, x_n)\). Since \(X_1\) must be orthogonal to \(X_1\) and all coordinates of \(X_1\) are nonnegative, without loss of generality, we may assume that \(x_1, x_2, \ldots, x_a\) are positive and \(x_{a+1}, \ldots, x_n\) are nonpositive. Let \(b = n - a\). Then, since \(\lambda_n(G)\) is the smallest eigenvalue of \(A(G)\), we have