THE PROPERTIES OF MATRICES $\alpha I + \beta J$
AND SOME RELATED PROBLEMS*

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Abstract

Let $F$ be a field, $v(\geq 2)$ an integer, $I_n$ the identity matrix of order $v$, and $J_n$ the all-one matrix of order $n$. Let $\mathcal{D} = \{ \alpha I_n + \beta J_n \in GL_n(F) | \alpha, \beta \in F \}$, and $\mathcal{C} = \{ C \in GL_n(F) | \text{the row-sums and the column-sums of } C \text{ are all equal} \}$. In this paper, we study (1) the center and the centralizer of $\mathcal{C}$, (2) the centralizer and the normalizer of $\mathcal{D}$ when $v=2$, and (3) the orders of $\mathcal{D}$, $\mathcal{C}$ and the normalizer of $\mathcal{D}$.

In this paper, we study (1) the center and the centralizer of $\mathcal{C}$ when $v\geq 2$, (2) the centralizer and the normalizer of $\mathcal{D}$ when $v=2$ (Although the corresponding block designs are trivial, this property has its own right to be investigated), and (3) the orders of $\mathcal{D}$, $\mathcal{C}$ and the normalizer of $\mathcal{D}$ when $F$ is a finite field.

§ 1.

When $v \geq 2$, for the relationship between $\mathcal{D}$ and $\mathcal{C}$, we have

Theorem 1. $\mathcal{D}$ is the center of $\mathcal{C}$ as well as the centralizer of $\mathcal{C}$.

Proof. First suppose that $v > 2$. From Theorem 2 in [3], we know that $\mathcal{D}$ is included in the centralizer of $\mathcal{C}$. Now suppose that $A = (a_{ij})_{v \times v}$ is an element of the centralizer of $\mathcal{C}$. Set

$$K = (k_{ij})_{v \times v} = \begin{pmatrix} 0 & I_{v-1} & 0 \\ 1 & 0 & 1 \\ 1 & v-1 & 1 \end{pmatrix}$$

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Obviously, \( K \in \mathcal{G} \). Then \( KA = AK \), that is,

\[
\begin{align*}
\alpha_{ij} &= \alpha_{i-1,j-1} \quad ( \delta \neq 1, j \neq 1 ), \\
\alpha_{ij} &= \alpha_{i,j-1} \quad ( j \neq 1 ), \\
\alpha_{ij} &= \alpha_{i-1,j} \quad ( \delta \neq 1 ), \\
\alpha_{ij} &= \alpha_{ij}. \quad \text{(1)}
\end{align*}
\]

Therefore, \( A \) is a circular matrix of the form

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{v-1} & \alpha_v \\
\alpha_v & \alpha_1 & \cdots & \alpha_{v-2} & \alpha_{v-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_2 & \alpha_3 & \cdots & \alpha_v & \alpha_1 
\end{pmatrix}
\]

Set

\[
L = (l_{ij})_{v \times v} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 0 \\
\end{pmatrix}
\]

with \( l_{12} = l_{23} = l_{34} = \cdots = l_{v-1} = 1 \) and all the other \( l_{ij} = 0 \). Clearly, \( L \in \mathcal{G} \). Then \( LA = AL \), that is,

\[
(a_0, a_1, a_2, \ldots, a_{v-2}, a_{v-1}) = (a_2, a_1, a_0, \ldots, a_v, a_0). \quad \text{(2)}
\]

Set

\[
M = (m_{ij})_{v \times v} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 
\end{pmatrix}
\]

with \( m_{12} = m_{23} = m_{34} = \cdots = m_{v-1,2} = 1 \) and all the other \( m_{ij} = 0 \). Clearly, \( M \in \mathcal{G} \). Then \( MA = AM \), that is,

\[
(a_0, a_1, a_2, \ldots, a_{v-2}, a_{v-1}) = (a_0, a_1, a_0, \ldots, a_2, a_2). \quad \text{(3)}
\]

Combining (2) and (3), we have the equality relationship among \( a_1, a_2, \ldots, a_v \) shown as follows:

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{v-1} & \alpha_v \\
\end{array}
\]

that is,

\[
a_2 = a_3 = \cdots = a_v.
\]