BOUNDARY VALUE PROBLEMS
OF SINGULARLY PERTURBED
INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, the boundary value problem for the integro-differential equation with a small parameter $\varepsilon > 0$:

$$
\begin{align*}
\varepsilon^2 z'' &= f(t, T_1 z, \ldots, T_m z, z, \varepsilon), \\
\alpha_i z(i, \varepsilon) + (-1)^i \beta_i z'(i, \varepsilon) &= A_i(\varepsilon), \quad i = 0, 1
\end{align*}
$$

is discussed, where $T_i$'s are integral operators defined on $C[0,1]$:

$$
T_i: g(t) \rightarrow T_i g = \varphi_i(t, \varepsilon) + \int_0^t K_i(t, \xi, \varepsilon) g(\xi) \, d\xi.
$$

Using the differential inequality technique, the existence of solutions is proved and the estimate of solutions is obtained as well. In particular, this result applied to the high-order ($n \geq 3$) boundary value problem for ordinary differential equations with a small parameter $\varepsilon > 0$:

$$
\begin{align*}
\epsilon^2 y^{(n)} &= f(t, y, y', \ldots, y^{(n-2)}, \varepsilon), \\
y^{(j)}(0, \varepsilon) &= \alpha_j(\varepsilon), \quad j = 0, 1, \ldots, n-3, \\
\alpha y^{(n-2)}(i, \varepsilon) + (-1)^i \beta y^{(n-1)}(i, \varepsilon) &= A_i(\varepsilon), \quad i = 0, 1.
\end{align*}
$$

Key words. Integro-differential equation, singularly perturbation, boundary value problem

1. Introduction

One frequently encounters in applications boundary value problems for the singularly perturbed equation

$$
\varepsilon^2 x^{(n)} = f(t, x, \ldots, x^{(n-1)}, \varepsilon)
$$

(cf. for example, similarity solutions of the Navier-Stokes equations at high Reynolds number[1]). However, when $n \geq 3$, the investigation of these problems is rather difficult. Hence so far there are only a few works in this direction. The one earlier work was due to Wasow, who studied the existence of periodic solutions (cf. [2]). Howes[3],[4] first applied
the differential inequality technique to some singularly perturbed two-point boundary value problems for high order equations.

Since the theory of boundary value problems for the second order equations has been developed greatly, one natural idea is to transform the high order cases into the second order ones. To do this, we consider the following singular perturbation of the integro-differential equation

\[ \varepsilon^2 x'' = f(t, T_1 x, \cdots, T_m x, x, x', \varepsilon), \]

where \( T_i \)s \( (i = 1, \cdots, m) \) are some integral operators.

In [5], using this approach, we prove the existence and uniqueness of solutions to the singularly perturbed boundary value problems for third order equations. The aim of this paper is to extend further the above idea to the general case. Mainly we prove a general existence theorem on the singularly perturbed Robin boundary value problems for integro-differential equation:

\[ \varepsilon^2 x'' = f(t, T_1 x, \cdots, T_m x, x, \varepsilon), \]

\[ a_i x(i, \varepsilon) - (-1)^i \beta_i x(i, \varepsilon) = A_i(\varepsilon), \quad i = 0, 1, \]

where \( T_i \)s \( (i = 1, 2, \cdots, m) \) are some integral operators defined on \( C[0,1] \):

\[ T_i : g(t) \rightarrow T_ig = \phi_i(t, \varepsilon) + \int_0^t K_i(t, \xi, \varepsilon)g(\xi) \, d\xi, \]

and \( \alpha_i \geq 0, \beta_i \geq 0 \) are constants with \( \alpha_i + \beta_i \neq 0, \, i = 0, 1. \)

This treatment is convenient for the high order problem. Indeed, using this result, we prove the existence, uniqueness and asymptotic estimate of solutions to the following Robin boundary value problem

\[ \varepsilon^2 y^{(n)} = f(t, y, \cdots, y^{(n-2)}, \varepsilon), \]

\[ y^{(j)}(0, \varepsilon) = a_j(\varepsilon), \quad j = 0, 1, \cdots, n - 3, \]

\[ \varepsilon a_i y^{(n-2)}(i, \varepsilon) - (-1)^i \beta_i y^{(n-1)}(i, \varepsilon) = A_i(\varepsilon), \quad i = 0, 1, \]

under the only assumption \( f^{(n-2)} \geq \sigma > 0. \)

Of course, singularly perturbed boundary value problem for integro-differential equations is itself worth investigation (cf. [6]).

The plan of this paper is as follows. In Section 2, formal solutions of the problem are constructed. Our main results are proved in Section 3. Finally Section 4 exhibits an application of the main results to the high order problem.

## 2. Approximate Solutions

Let \( I = [0, 1], I_{e_0} = (0, e_0), \tilde{I}_{e_0} = [0, e_0] \), where \( e_0 \) is a small positive number. Assume that the following conditions hold:

(I) \( f(t, z_1, \cdots, z_m, x, \varepsilon) \in C^2(I \times \mathbb{R}^{m+1} \times \tilde{I}_{e_0}), \phi_i(t, \varepsilon) \in C^2(I \times \tilde{I}_{e_0}), K_i(t, \xi, \varepsilon) \in C^2(I \times I \times \tilde{I}_{e_0}) \) \( (i = 1, 2, \cdots, m) \), \( A_0(\varepsilon), A_1(\varepsilon) \in C^1(\tilde{I}_{e_0}) \) and

\[ f_x(t, z_1, \cdots, z_m, x, \varepsilon) > 0. \]  

(II) The equation

\[ 0 = f(t, T_1^0 x, \cdots, T_m^0 x, x, 0). \]