Numerical analysis of secondary and tertiary states of fluid flow and their stability properties

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Abstract. In fluid systems exhibiting a gradual transition to a turbulent state in dependence on external parameters the increasing complexity of motion is usually caused by sequences of bifurcations in the solution space. Through the consideration of the configuration of highest symmetry compatible with the basic physical problem most bifurcations can be identified by their symmetry breaking properties. Because it can incorporate all available symmetries in the selection of expansion functions, the Galerkin method turns out to be especially useful for the numerical analysis of highly symmetric fluid flow and their instabilities. By allowing for the broken symmetries, the nonlinear evolution of the flow can be followed numerically through several bifurcations. A complete stability analysis with respect to arbitrary infinitesimal disturbances is possible on the basis of Floquet's theory. This method has been applied in cases of Taylor–Couette flow, Rayleigh–Bénard convection, and shear flows with cubic profiles.

1. Introduction

Ever since the advent of electronic computers one of their major applications has been fluid dynamics. As speed and capacity of computers have increased computational fluid dynamics has developed into a field of its own approaching in importance the traditional fields of experimental and theoretical fluid dynamics. In fact, to a certain extent computational fluid dynamics is replacing laboratory measurements of fluid motions since many flow configurations can be analyzed more readily and with less expense on computers than in a laboratory setting. Computational simulations of flows allow us to present a far more detailed picture of complex motion than is usually possible with laboratory visualization methods. With the growth of information provided by the computer, however, increasing need is felt for tools with which the information can be processed and interpreted.

The need for an improved understanding of complex flows is especially urgent in the case of turbulent flows where the importance of coherent structures has become recognized in recent years. Typically, these structures reflect the mechanisms of instability which are responsible for the transition to turbulence. Often several transitions can be recognized in flows for which the onset of turbulence occurs over an extended regime of the Reynolds number. The study of these transitions is thus not only of interest for the initial stages in the development of turbulence, but also can provide an understanding of the mechanisms involved in the generation of typical coherent structures.

The analysis of instabilities and transitions naturally starts at the simplest configuration that exhibits the particular mechanism of instability. The attention of
Fluid dynamicists has been focused for this reason on highly symmetric configurations such as flow in a two-dimensional channel or the flow between coaxial differentially rotating cylinders. The simplicity and high degree of symmetry of these examples is not only attractive from the theoretical point of view, but also convenient in computational respects. For the purpose of the study of transitions, high degrees of symmetry are even more important and often they are essential. Most instabilities can be readily recognized only by their symmetry breaking properties. Without the presence of symmetry, transitions tend to occur inhomogeneously and in a gradual way as a function of external parameters. In the mathematical terminology 'bifurcations' become replaced by 'imperfect bifurcations'. The aim of this paper is to present some typical examples for the use of symmetries in the numerical investigation of transitions leading to complex forms of fluid flow.

In Section 2 we shall introduce the basic equations governing the flow between differentially rotating coaxial cylinders. Since we allow for different temperatures $T_1$ and $T_2$ prescribed at the inner and outer cylinders, respectively, and since we permit the direction of gravity to vary from the axial direction to the radial direction as suggested by the addition of the centrifugal force in typical laboratory experiments, we include in our formulation many cases of special interest. Besides the Taylor–Couette system corresponding to the isothermal case, we find as a special case the vertical layer heated from the side when the cylinders are at rest and gravity is pointing in the axial direction. The small gap limit will be used in all cases in order to take advantage of the plane layer geometry. When gravity points in the radial direction we recapture the Rayleigh–Bénard problem of convection in a layer heated from below. All of these three special cases have in common that spatial degrees of freedom motion are occupied in a sequence of bifurcations starting with simple roll-like flows and leading to turbulent fluid motions. In Section 3 we shall briefly outline the method of analysis that has been used in the past for these cases and refer to the literature for detailed results. A particular advantage of the Galerkin procedure described here is the combination with a stability analysis which answers the question of the physical realizability of the numerically obtained solution. Some general aspects are discussed in the concluding section of the paper.

2. Some basic mechanisms of flow instability

We consider the annular region between two vertical coaxial cylinders with radii $R_1$, $R_2$ which are rotating with the angular velocities $\Omega_1$ and $\Omega_2$. It is assumed that the inner cylinder with radius $R_1$ is kept at the temperature $T_1$, while the outer cylinder is kept at the higher temperature $T_2$. Using the gap width $d = R_2 - R_1$ as length scale, $d^2/v$ as time scale where $v$ is the kinematic viscosity and $(T_2 - T_1)/P$ as scale of the temperature, we write the dimensionless Navier–Stokes equation for the velocity vector $\mathbf{u}$ and the heat equation for the deviation $\Theta$ of the temperature from the