PROBABILISTIC APPROACH TO THE DIRICHLET PROBLEM OF SECOND ORDER ELLIPTIC PDE\footnote{Received October 12, 1987.}

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Abstract

In this paper we provide a probabilistic approach to the following Dirichlet problem

\[
\begin{aligned}
\left( \sum \frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial}{\partial x^j} \right) + \sum b^{i} \frac{\partial}{\partial x^i} \right) u &= 0, \quad \text{in } D, \\
\sigma &= g, \quad \text{on } \partial D,
\end{aligned}
\]

without assuming that the eigenvalues of the operator

\[
\sum \frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial}{\partial x^j} \right) + \sum b^{i} \frac{\partial}{\partial x^i} + \xi
\]

with Dirichlet boundary conditions are all strictly negative. The results of this paper generalize those of Ma\cite{10}.

In 1981, Chung, and Rao\cite{20} solved the following Dirichlet problem

\[
\begin{aligned}
\left( \frac{1}{2} \Delta + \xi \right) u &= 0, \quad \text{in } D, \\
u &= g, \quad \text{on } \partial D,
\end{aligned}
\]

by probabilistic method under the assumption that the eigenvalues of the Schrödinger operator \( \left( \frac{1}{2} \Delta + \xi \right) \) with Dirichlet boundary conditions are all strictly negative. Ma\cite{10} removed the above assumption and completely solved the Dirichlet problem (0.1) by using probabilistic method.

The method used in Ma\cite{10} also applies to the following Dirichlet problem

\[
\begin{aligned}
\left( L + \xi \right) u + g_2 &= 0, \quad \text{in } D, \\
u &= g_2, \quad \text{on } \partial D,
\end{aligned}
\]

when \( L \) is a symmetric, strongly elliptic operator of the form \( \sum \frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial}{\partial x^j} \right) \).

However, the method of Ma\cite{10} fails when \( L \) is nonsymmetric. Up to now, no suitable method has been found to treat the problem (0.2) for the case where \( L \) is nonsymmetric. The present paper is aimed to fill this gap.

More precisely, this paper is devoted to find a probabilistic method to solve the following Dirichlet problem

\[
\begin{aligned}
\left( L + \xi \right) u + g_2 &= 0, \quad \text{in } D, \\
u &= g_2, \quad \text{on } \partial D,
\end{aligned}
\]
where $D$ is a bounded $C^{2,\alpha}(0<\alpha<1)$ domain of $\mathbb{R}^d$, $\xi$ and $g_1$ are real-valued H"older continuous functions defined in $D$, $g_2$ is a real-valued continuous function defined on $\partial D$ and $L$ is an operator of the following form:

$$L = \sum \frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial}{\partial x^j} \right) + \sum \delta^i - \frac{\partial}{\partial x^i},$$

with its coefficients satisfying the following assumptions:

(A1) all the coefficients are real-valued functions;
(A2) $(a^{ij})$ is symmetric and there exists a positive constant $M$ such that for any $x, y \in \mathbb{R}^d$,

$$\sum a^{ij}(x) y^j > M |y|^2;$$

(A3) $a^{ij} \in C^1_2(\mathbb{R}^d) = \{ f \in C^2(\mathbb{R}^d) : \text{for any } 0<|a|<2, D^2 f \text{ is bounded} \};$
(A4) $b^i \in C^1_2(\mathbb{R}^d)$. Under the above assumptions it is known (see [8]) that there exists a unique $d$-dimensional diffusion process $(\{X_t\}_{t \geq 0}, \{P_t\}_{t \in \mathbb{R}^d})$ such that for every $f \in C^2_0(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$,

$$f(x_t) - f(x_0) - \int_0^1 (L f)(X_s) \, ds$$

is a $P^*-$martingale.

It is also known (see [4]) that the diffusion process $X_t$ has a transition density function $p(t, x, y)$ which satisfies the following conditions:

(i) $p$ and $\frac{\partial p}{\partial t}$ are continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$;
(ii) $D^2 p(|a| < 2)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with

$$D^2 p(t, x, y) \leq Ct^{-n} \Gamma(|a|, x, y), \quad |a| = 0, 1,$$

where $C$ and $\lambda$ are positive constants, $\Gamma$ is the transition density function of the $d$-dimensional Brownian motion;

(iii) for any fixed $y \in \mathbb{R}^d$, $p(\cdot, \cdot, y)$ is a solution of the following equation

$$\frac{\partial p}{\partial t} = L p$$

with the initial condition

$$p(t, x, y) \rightarrow \delta(x-y) \quad \text{as } t \downarrow 0,$$

which means that for every bounded continuous function $f$ in $\mathbb{R}^d$ we have

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy = f(x).$$

The contents of this paper are organized as follows. In Section 1 we determine the explicit expressions of the generators of the semigroup $T_t$ of the killed diffusion of $X_t$ and the Feynman-Kac semigroup $F_t$ defined by

$$F_t f(x) = E^x \left[ \exp \left( \int_0^t \xi(X_s) \, ds \right) f(X_t), \ t < \tau_D \right].$$

In Section 2 we use the results of Section 1 to provide a probabilistic treatment for the Dirichlet problem (0.3).