ON THE BOUNDEDNESS AND THE STABILITY PROPERTIES OF SOLUTION OF CERTAIN FOURTH ORDER DIFFERENTIAL EQUATIONS

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Abstract

This paper investigates equation (1) in two cases: (i) \( P = 0 \), (ii) \( P(x, y, z, w) \leq (A + |y| + |z| + |w|)q(t) \), where \( q(t) \) is a nonnegative function of \( t \).

For case (i) the asymptotic stability in the large of the trivial solution \( x = 0 \) is investigated and for case (ii) the boundedness result is obtained for solutions of equation (1). These results improve and include several well-known results.

Key words nonlinear differential equations of the four order, boundedness, stability

Consider the equation

\[ \ddot{x} + \varphi(z)\dot{z} + f(z) + g(x) + h(x) = P(t, x, \dot{x}, \dot{z}, \dot{w}) \]  

where \( \varphi(z) \), \( f(z) \), \( g'(x) \), \( h'(x) \) and \( P(t, x, \dot{x}, \dot{z}, \dot{w}) \) are continuous functions for the arguments displayed explicitly, and are such that the existence and uniqueness of solutions, as well as their continuous dependence on the initial conditions, are guaranteed.

Equation (1) has an equivalent system

\[ \begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= w,
\end{align*} \]

\[ \varphi = P(t, x, y, z, w) - h(x) - g(y) - f(z) - \varphi(z)w \]

Set

\[ \begin{align*}
\varphi_1(z) &= \frac{1}{z} \int_{s}^{z} \varphi(s)ds \quad (z \neq 0), \quad \varphi_1(0) = \varphi(0); \\
f_1(z) &= \frac{f(z)}{z} \quad (z \neq 0), \quad f_1(0) = f'(0), \\
g_1(y) &= \frac{g(y)}{y} \quad (y \neq 0), \quad g_1(0) = g'(0).
\end{align*} \]
In the case $P=0$ we have

**Theorem 1** Suppose the following conditions are satisfied:

(i) $f(0)=g(0)=h(0)=0$.

(ii) there are positive constants $a,b,c,d$, and $\delta$ such that

\[
abc-cg'(y)-a\varphi(z)\geq \delta > 0,
\]

for all $y$ and $z$.

(iii) $d - \frac{a\delta}{4c} < h'(x) < d$, for all $x$ and

\[
h(x)\sgn x \to +\infty, \quad |x| \to \infty,
\]

(iv) $0 \leq g_1(y) - c < \frac{\delta}{8c} \sqrt{\frac{d}{2ac}}$, for all $y$,

(v) $0 \leq f_1(z) - b < \frac{c^3}{2d}$, for all $z$,

where $0 < a < b < \frac{\delta}{2acD}$, $D = ab + \frac{bc}{d}$;

(vi) $\varphi(z) > a$, $\varphi_1(z) - \varphi(z) < \frac{\delta}{2b^3c}$, for all $z$.

Then the trivial solution of system (2) is asymptotically stable in the large.

**Remark 1** From conditions (ii), (iv) and (vi) we can obtain

\[
g'(y) < ab, \quad \varphi(z) < \frac{bc}{d}.
\]

**Remark 2** When $\varphi(z) \equiv a, \: f(x) = bx, \: g(y) = cy, \: h(x) = dx$, equation (1) reduces to the linear constant coefficient differential equation

\[
\ddot{x} + a\dot{x} + b\dot{z} + c\dot{z} + dx = 0
\]

and conditions (i) − (vi) of Theorem 1 reduce to the corresponding Routh-Hurwitz criterion.

**Remark 3** Theorem 1 improves and includes Theorem 6.6 and Theorem 6.9 of [1] and the results of [3], [4] and [5] are also special cases of our result.

In the case $P(t, x, y, z, w) \equiv 0$ we have

**Theorem 2** Suppose the following conditions are satisfied:

(i) $f(0)=g(0)=0$.

(ii) conditions (ii) − (vi) of Theorem 1 hold,

(iii) $\left| P(t, x, y, z, w) \right| \leq (A + |y| + |z| + |w|)q(t) = 0$,

where $q(t)$ is a nonnegative and continuous function of $t$, and satisfies $\int_0^t q(s)ds \leq B < \infty$, for all $t \geq 0$, $A$ and $B$ are positive constants. Then for any given finite $x_0, y_0, z_0, w_0$, there exists constant $K = K(x_0, y_0, z_0, w_0)$, such that any solution $x(t), y(t), z(t), w(t)$, of system (2) determined by

\[
x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad w(0) = w_0
\]

satisfies for all $t \geq 0$,

\[
|x(t)| \leq K, \quad |y(t)| \leq K, \quad |z(t)| \leq K, \quad |w(t)| \leq K.
\]