DISCONTINUOUS AND IMPULSIVE EXCITATION

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Abstract

In this paper, we study the solution of differential equation with Dirac function and Heaviside function, arising from discontinuous and impulsive excitation. Firstly, according to the theory of differential equation, we suggest \( x(t) = x_1(t) + x_2(t)H(t-a) \); then we derive the equation of \( x_1(t) \) and \( x_2(t) \) by terms of property of distribution, and by solving \( x_1(t) \) and \( x_2(t) \) we obtain \( x(t) \); finally, we make a thorough investigation about periodic impulsive parametric excitation.

I. Introduction

In nonlinear oscillation field, the governing equation becomes a differential equation containing Dirac function or Heaviside function, arising from impulsive or discontinuous excitation. To solve this kind of equation is quite difficult, especially in nonlinear case.

When forced excitation is impulse, second order linear differential equation can be solved by theorem of impulse. This result was developed to deal with higher order linear differential equation in article [1]. By means of approximate property of Dirac function and the ideal of Poincaré map, prof. Hsu dealt with the problem of periodic impulsive parametric excitation in articles [2-5]. In this paper, we intend to discuss the impulse and discontinuity of forced excitation and parametric excitation. The main tools are theory of O.D.E, the property of distribution, singular perturbation methods and ideal of Poincaré's map. Firstly we suggest \( x(t) = x_1(t) + x_2(t)H(t-a) \) and derive the equation of \( x_1(t) \) and \( x_2(t) \); by solving \( x_1(t) \) and \( x_2(t) \), we obtain \( x(t) \). Finally we make a thorough investigation about periodic impulsive parametric excitation and point out that present results are only particular results, and the correct answer is in indefinite form.

In nonlinear oscillation problems, the governing equation becomes the differential equation containing \( \delta(t-a) \) or \( H(t-a) \), if impulse or discontinuity happens at \( t=a \). We can reduce this kind of differential equation to first order equation system. Obviously, coefficient term or nonhomogeneous term of the equation system is a continuous function except \( t=a \). According to the theorems of existence and extension, we can consider that the solution of the equation exists in \( C^1 \) space except \( t=a \), therefore we can suggest that the solution form is

\[
x(t) = x_1(t) + x_2(t)H(t-a)
\]

(1.1) is a development of Calatheórdory solution in \( C \) space. If \( x_2(a) = 0 \), (1.1) is a general Caratheórdory solution, otherwise \( x_2(a) \neq 0 \) (1.1) is a kind of solution with discontinuous property.

It is obvious that we will encounter \( H(t-a) \) and \( \delta(t-a) \) in the course of solving (1.1),
hence we give two properties of this kind of distribution.

**Lemma 1** In the space of distribution, we have

1. $H(x) = H^2(x) = \cdots = H^n(x)$ \hspace{1cm} ($n$ = arbitrary integer)

where $H(x)$ is Heaviside function.

2. If $f(x)$ is a sufficiently smooth function, then

$$
\int_0^1 f(x) \delta^{(n)}(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!} f^{(n-k)}(0) \delta^{(k)}(x)
$$

where $n$ is an arbitrary integer, $\delta(x)$ is Dirac function.

**Lemma 2** In the space of distribution, $\delta(x)$ is Dirac function, then $H(x), \delta(x), \delta'(x), \cdots, \delta^{(n)}(x)$ are linearly independent of each other.

**II. The Forced Excitation Is Impulse or Discontinuity**

When the forced excitation is impulse, the governing equation is reduced to

$$
\dot{x} = Ax + ef(x) + \lambda \delta(t-a)
$$

(2.1)

where $x$ and $\lambda$ are $n$-dimension vectors, $f(x)$ is an $n$-dimension polynomial function vector with power more than one, $A$ is an $n \times n$ constant matrix, $0 < \varepsilon \ll 1$ is a small parameter.

The solution of (2.1) is

$$
x(t) = x_1(t) + x_2(t) H(t-a)
$$

(2.2)

where $x_1(t)$ satisfies the equation

$$
\ddot{x}_1(t) = Ax_1(t) + ef(x)
$$

(2.3)

and the definite conditions of $x$. If a uniformly valid asymptotic solution is found by singular perturbation method for $\varepsilon > 0$ small enough,

$$
x_1(t) = \bar{x}_1(t)
$$

(2.4)

then $x_2(t)$ should satisfy the equation

$$
\ddot{x}_2(t) H(t-a) + x_2(t) \delta(t-a) = Ax_1(t) H(t-a) + ef(\bar{x}_1(t), x_2(t)) + \lambda \delta(t-a)
$$

(2.5)

According to Lemmas 1 and 2, we have

$$
\ddot{x}_2(t) = Ax_2(t) + eg(\bar{x}_1(t), x_2(t)), \hspace{0.5cm} x_2(a) = \lambda
$$

(2.6)

By means of singular perturbation method or numerical method, we obtain

$$
x_2(t) = \bar{x}_2(t)
$$

(2.7)

Thus the solution of (2.1) is

$$
x(t) = \bar{x}_1(t) + \bar{x}_2(t) H(t-a)
$$

(2.8)

Now, we consider that the forced excitation is discontinuity, namely the equation is

$$
\dot{x} = Ax + ef(x) + F(t)
$$

(2.9)

where $F(t)$ is a discontinuity at $t = a$

$$
F(t) = \begin{cases} 
\mathcal{P}_1(t) & (0 < t \leq a) \\
\mathcal{P}_2(t) & (t > a) 
\end{cases}
$$

(2.10)