APPLICABILITY OF HAMILTON-JACOBI METHOD TO NONLINEAR NONHOLONOMIC SYSTEMS

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Abstract

This paper uses Poincaré formalism to obtain a generalization of the Hamilton-Jacobi method of integrating dynamical systems moving with nonlinear nonholonomic constraints. Necessary and sufficient conditions are investigated for the applicability of this method to such systems. The method is illustrated by considering some concrete examples of nonholonomic systems.

I. Introduction

Consider a dynamical system with independent coordinates $x_1, \ldots, x_n$, kinetic energy $T$ and force function $U$. Let the system be subjected to $(n-m)$ nonlinear nonholonomic constraints of the form

$$f_p(x_1, \eta_1, t) = 0, \quad p = 1, 2, \ldots, n, \quad \alpha = m + 1, \ldots, n \quad (1.1)$$

Here $\eta_1, \ldots, \eta_n$ are the Poincaré parameters. If in a virtual displacement the independent parameters, $\omega_1, \ldots, \omega_n$ satisfy the relations $^{(1)}$:

$$\frac{\partial f_p}{\partial \eta_p} \omega_p = 0 \quad (1.2)$$

the motion of the system is determined by equations $^{(1)}$:

$$\frac{d}{dt} \frac{\partial T}{\partial \eta_p} - C_{pq} \frac{\partial T}{\partial \eta_q} - C_{pq} \frac{\partial T}{\partial \eta_q} - X_p(T + U) = \mu_p \frac{\partial f_p}{\partial \eta_p} \quad (1.3)$$

$$(p, q, r = 1, 2, \ldots, n, \quad \alpha = m + 1, \ldots, n)$$

where $\mu_1, \ldots, \mu_n$ are undetermined multipliers and repeated indices denote summation. The infinitesimal displacement operators

$$X_0 = \frac{\partial}{\partial t} + \xi^i(x, t) \frac{\partial}{\partial x_i}, \quad X_p = \xi^i_p(x, t) \frac{\partial}{\partial x_i}$$

form a group, yielding the commutation relations

$$(X_0, X_p) = C_{pq} X_q, \quad (X_p, X_q) = C_{pq} X_q$$

which define the quantities $C_{pq}$ as functions of the $x$'s and $t$.

In deriving equations (1.3), the variation $dG(\delta G)$ of an arbitrary function $G(x, t)$ in a real (virtual) displacement of the system is defined by
By means of (1.4), equations (1.3) may be transformed into the canonical form:

\[ \eta_t = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -X_j H + C_{i}^{j} y_i + C_{i}^{j} \eta_j y_i + \mu_x \frac{\partial f_x}{\partial \eta_x} \]  

(1.5)

where the new variables \( y_j \) are given by

\[ y_j = \frac{\partial T}{\partial \eta_j} \]  

(1.6)

and \( H(x, y, t) \) is the Hamiltonian function defined by

\[ H(x, y, t) = y_j \eta_j - (T + U) \]  

(1.7)

The aim of this paper is to give a formulation of the Hamilton-Jacobi method so as to cope with the problem of integration of equations (1.5) which describe the motion of the nonlinear nonholonomic system with constraints (1.1). In a recent series of papers [6–8], Van Dooren has studied this problem in terms of Lagrangian coordinates when the equations of nonlinear nonholonomic constraints are homogeneous in the generalized velocities. However, his generalization of the Hamilton-Jacobi method is valid under certain conditions which are discussed in [5] by Rumyantsev and Sumbatov. Very recently, Ghoril2r has used Poincaré formalism to discuss this problem for dynamical systems which are subjected to linear nonholonomic constraints. Here we shall extend this work when the equations of nonholonomic constraints are neither linear nor homogeneous in the Poincaré parameters.

II. Hamilton-Jacobi Method

We consider the Hamilton function of action with variable upper limit, namely,

\[ S = \int_{t_0}^{t} L(x, \eta, t) \, dt \]  

(2.1)

where \( L = T + U \) is the Lagrangian of the dynamical system. The functional (2.1), in view of relation (1.7), becomes

\[ S = \int_{t_0}^{t} [y_j \eta_j - H(x, y, t)] \, dt \]  

(2.2)

Taking the \( \delta \)-variation of (2.2), we find that

\[ \delta S = \int_{t_0}^{t} \left( y_j \delta \eta_j + \eta_j \delta y_j - \omega_x X_j H - \frac{\partial H}{\partial \eta_j} \delta y_j \right) \, dt \]

We know that \( \delta \eta_j \) satisfy the relations

\[ \delta \eta_j = \frac{d\omega_x}{dt} + C_{i}^{j} \omega_x + C_{i}^{j} \omega_y \eta_i \]  

(2.3)

In view of equations (1.2), (1.5) and (2.3), we obtain

\[ \delta S = \int_{t_0}^{t} \frac{d}{dt} (y_j \eta_j) \, dt \]