A UNIFORMLY CONVERGENT SECOND ORDER DIFFERENCE SCHEME
FOR A SINGULARLY PERTURBED SELF-ADJOINT ORDINARY
DIFFERENTIAL EQUATION IN CONSERVATION FORM

Guo Wen (郭 文) Lin Peng-cheng (林鹏程)
(Fuzhou University, Fuzhou)
(Communicated by Lin Zong-chi; Received June 3, 1987)

Abstract
In this paper, based on the idea of El-Mistikawy and Werle, we construct a difference scheme for a singularly perturbed self-adjoint ordinary differential equation in conservation form. We prove that it is a uniformly convergent second order scheme.

I. Introduction

Consider the following singular perturbation self-adjoint problem in conservation form:

\[ L(u(x)) = -e(p(x)u')' + q(x)u = f(x) \quad (x \in (0, 1)) \]
\[ u(0) = a_0, \quad u(1) = a_1 \]

where \( e > 0 \) is a small parameter; \( a_0 \) and \( a_1 \) are given constants; \( p, q \) and \( f \) are sufficiently differentiable for our purpose, and

\[ p(x) \geq a > 0, \quad q(x) \geq \beta > 0 \quad (x \in [0, 1]) \]

Hegarty et al. derived a second accurate difference scheme for (1.1) in the situation \( p \equiv 1 \). We proved their result in a different way from [2] (see [3]). Here we construct a difference scheme for (1.1), based on the idea of El-Mistikawy and Werle. Starting from a decomposition of the solution of the differential equation, we estimate the truncation error for the constructed scheme. Then we prove that the scheme is a second order one uniformly convergent in \( e \).

Throughout the paper we let \( c \) denote positive constants that may take different values in different formulas, but that are always independent of \( h \) and \( e \). And the estimations of orders are also made in the sense that is independent of \( h \) and \( e \).

II. Derivation of the Scheme

Reduce (1.1) to the equivalent form:

\[ -eu' + ea(x)u' + b(x)u = F(x) \quad (x \in (0, 1)) \]
\[ u(0) = a_0, \quad u(1) = a_1 \]

where \( a(x) = p'(x)/p(x) \), \( b(x) = q(x)/p(x) \) and \( F(x) = f(x)/p(x) \). Let \( J \) be a positive integer and define the uniform mesh length \( h = 1/J \), and mesh points \( x_j = jh \). Consider the problem:
\[ \begin{align*}
L_x U & = -e U'' - e a U' + b U = F & \quad (x \in (x_{j-1}, x_{j+1})) \\
U(x_{j-1}) &= u_{j-1}, \quad U(x_{j+1}) = u_{j+1} & \quad (j = 1, \ldots, J - 1)
\end{align*} \]

where \( u_j \) is an approximation of \( u(x_j) \); \( a, b \) and \( F \) are some approximations of \( a(x), b(x) \) and \( F(x) \) respectively. By the method of El-Mistikawy and Werle, we take

\[ b(x) = \begin{cases} 
\frac{1}{2} (b_{j-1} + b_j) \triangleq b^- & \text{when } x \in [x_{j-1}, x_j) \\
\frac{1}{2} (b_j + b_{j+1}) \triangleq b^+ & \text{when } x \in (x_j, x_{j+1})
\end{cases} \quad (2.2) \]

where \( b_j = b(x_j) \). We shall also write \( F_j \) etc. \( F(x) \) is similarly defined. But for \( a(x) \), since there is a factor \( p'(x) \), the definition is inadequate. For our purpose, we define

\[ a(x) = \begin{cases} 
\frac{1}{2} \left[ \frac{p_j - p_{j-1/2}}{0.5 h} \right] / p_{j-1} + \left( \frac{p_j - p_{j-1/2}}{0.5 h} \right) / p_j \triangleq a^- & \text{when } x \in [x_{j-1}, x_j) \\
\frac{1}{2} \left[ \frac{p_{j+1/2} - p_j}{0.5 h} \right] / p_{j-1} + \left( \frac{p_{j+1/2} - p_{j-1/2}}{0.5 h} \right) / p_{j+1} \triangleq a^+ & \text{when } x \in (x_j, x_{j+1})
\end{cases} \quad (2.3) \]

By the continuity of \( U'(x) \) at \( x_j \), we obtain the system of linear equations in \( u_j \):

\[ \begin{align*}
-\frac{e}{h^2} (r_j^- u_{j-1} + r_j^+ u_j + r_j^+ u_{j+1}) \\
= q_j^- F_j - q_j^+ F_j + q_j^+ F_{j+1} & \quad (1 \leq j \leq J - 1) \\
u_0 = a_0, \quad u_J = a_J
\end{align*} \quad (2.4) \]

This is the difference scheme we want, in which

\[ r_j^- = \frac{(n_1 - k_j) \exp(n_1)}{\exp(n_1 - k_j) - 1}, \quad r_j^+ = \frac{(n_2 - k_2) \exp(-k_2)}{\exp(n_2 - k_2) - 1} \quad (2.5) \]

\[ r_1 = -n_1 - \frac{n_1 - k_1}{\exp(n_1 - k_1) - 1}, \quad r_2 = k_2 - \frac{n_2 - k_2}{\exp(n_2 - k_2) - 1} \quad (r_j^+ = r_1 + r_2) \quad (2.6) \]

\[ \begin{align*}
q_j^- = & \frac{e}{2h^2 b^-} [n_1 \exp(n_1) (1 - \exp(-k_1)) \\
& - k_1 (\exp(n_1) - 1)]/[1 - \exp(n_1 - k_1)] \\
q_j^+ = & \frac{e}{2h^2 b^+} [n_2 (1 - \exp(-k_2)) \\
& + k_2 \exp(-k_2) (\exp(n_2) - 1)]/[1 - \exp(n_2 - k_2)] \\
q_j^+ = & q_j^- + q_j^+
\end{align*} \quad (2.7) \]

where

\[ \begin{align*}
n_1 = & \frac{a^- h}{2} - \frac{h}{\sqrt{e}} \sqrt{b^- + \frac{e(a^-)^2}{4}} \\
n_2 = & \frac{a^+ h}{2} - \frac{h}{\sqrt{e}} \sqrt{b^+ + \frac{e(a^+)^2}{4}}
\end{align*} \]