A PENALTY-HYBRID FINITE ELEMENT ANALYSIS
OF STOKES FLOW*

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Abstract
A type of penalty-hybrid variational principle is suggested for the analysis of
Stokesian flow. On such a basis, a finite element model is formulated featuring, among
others, a priori satisfaction of the deviatoric stress and hydrostatic pressure on linear
momentum balance equations. Also in the present scheme the hydrostatic pressure is
successfully eliminated at the element level, leaving only nodal velocities as solution
unknowns. A series of 4-node and 8-node quadrilateral elements are derived and examined.
Numerical examples demonstrating their characteristic behaviors are also included.

I. Introduction

The Stokes equations are the linear and stationary particularizations of the Navier-Stokes
equations, and identical in form to the equations of classical isotropic incompressible elasticity
timey which govern the linear response of rubber-like materials, solid rocket propellants, etc. The
Stokes equations are of general and typical research significance. It is therefore no wonder that, in
the recent twenty years, the finite element analysis of Stokes flow has been one among the areas that
immensely focuses the research interests of diverse investigators. In this aspect, according to the
ways the incompressibility constraint is dealt with, various types of formulation methods and
approaches emerged.

The stream function formulation and stream function-vorticity formulation\[1,2\] identically
satisfy the incompressibility constraint by the introduction of stream function. Thus there is no
problem with the continuity equation. However, inherent drawback hinders the imposing of
boundary conditions and handicaps the extending of the formulations to cover three-dimensional
cases.

Based on the Lagrange multiplier manipulation, the velocity-pressure\[3,4\] features the use of
hydrostatic pressure as the multiplier for the introduction of the incompressibility constraint. It was
recognized fairly early that the velocity and pressure approximations should satisfy certain
consistency condition, often known as the BB condition\[5,6\], in order to obtain convergent stable
solutions. Unfortunately, many simple and natural elements, such as those with equal-order
interpolations, fail to satisfy the BB condition, leading to the emergence of spurious pressure
modes\[7\]. Although there are a handful of convergent elements, they are inconvenient to be
implemented.

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In the reduced integration-penalty approach\cite{8,9}, the exterior penalty method is invoked to deal with the incompressibility constraint. In addition, reduced integration is enforced for the evaluation of penalty terms such that solution “locking” can be avoided. The method is remarkable in having the hydrostatic pressure eliminated at element level, thus uncoupling the momentum and continuity equations and reducing the system of the finite element equations. However, to obtain convergent stable solutions, the velocity space and the related pressure space are required to satisfy the BB condition. As reported in\cite{10}, \cite{11} and \cite{12}, many elements appear unstable. To attain stability, the order of integration should be excessively lowered, which is an expensive trade-off paid by the deterioration of both solution accuracy and convergence rate.

Similarly, the incompressibility constraint is introduced in the hybrid finite element model\cite{13,14} via the use of Lagrange multiplier-the hydrostatic pressure. At the same time, both the deviatoric stress tensor and the hydrostatic pressure are required to satisfy the linear momentum balance equations a-priori. Investigations reveals that the hybrid finite element method is capable of providing remarkable numerical results both in velocity and in pressure. However, because of the uncoupled nature of the incompressibility constraint, it is practically inevitable that nodal velocities and the constant terms, which are related to the arbitrarily varying pressure field over each element, appear simultaneously as solution unknowns in the finite element matrix equations.

With these backgrounds in view, a kind of penalty-hybrid variational principle is suggested in the present study for the analysis of Stokes flow. The present work features specifically the replacement of the uncoupled incompressibility constraint with the perturbed continuity equation. In addition, the deviatoric stress tensor and the hydrostatic pressure are required to satisfy the linear momentum balance equations a-priori. It can be ascertained that finite element model formulated on such basis includes only nodal velocities as solution unknowns with the hydrostatic pressure eliminated at element level. To illustrate the concepts, a series of 4-node and 8-node quadrilateral elements are derived and examined, their characteristic behaviors being demonstrated.

II. Stokes Flow and Analysis

Denoting $\Omega$ as an open bounded domain in $\mathbb{R}^d$ (where $d$ is the number of the dimension of the space), with piece-wise smooth boundary $\partial \Omega$ composed of disjoint subsets $S_u$ and $S_\sigma$, one has the Stokesian flow equations written as

$$\sigma' \cdot \nabla - \nabla p + \rho F = 0 \quad \text{in } \Omega \tag{2.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \tag{2.2}$$

$$D = (u \nabla + \nabla u)/2 \quad \text{in } \Omega \tag{2.3}$$

$$\sigma = -p I + \sigma', \quad \sigma' = 2\mu D \tag{2.4}$$

and boundary conditions:

$$u = \bar{u} \quad \text{on } S_u \tag{2.5}$$

$$T = \bar{T} \quad \text{on } S_\sigma \tag{2.6}$$

where $u$ is the velocity vector, $\sigma'$ the deviatoric stress tensor, $\sigma$ the Cauchy stress tensor, $p$ the hydrostatic pressure, $D$ the strain-rate tensor, $\rho$ the mass density, $\bar{u}$ the prescribed velocity on $S_u$, $\bar{T}$ the prescribed traction on $S_\sigma$, $F$ the prescribed body force per unit mass, $\mu$ the viscosity, and $\nabla$ the Hamilton operator.

By deviding domain $\Omega$ into a finite number of finite elements, i.e.