WHY WE HAVE NOT YET FOUND A RETROGRADE PLANET IN THE SOLAR SYSTEM?

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Abstract

In this paper we use the Jacobian integral of the circular restricted three-body problem to establish a testing function of a moving testing particle when it moves like a planet. This function determines whether or not the particle will stay in a definite region (which may be called "stable region", SR). By means of checking with an electronic computer, we can find that the SR of quasicircular orbit of retrograde planet motion is much less than the SR of direct planet motion. It is the reason why the existence of a retrograde planet is very rare.

I. Jacobian Integral of Circular Restricted Three-Body Problem

Except for the Mercury and the pluto, all the planets of the solar system have a small eccentricity $e$ and a small inclination $i$ to ecliptic plane. Therefore their orbits may be regarded as concentrical circles on the same plane. Imagine that a particle $q$ with a negligible mass $m$ passed by a planet $P$. Then the particle is in the gravitational field of the sun and the planet $P$ and its motion is a restricted three-body problem. When the orbit of the planet $P$ is a circle, there is a Jacobian integral as

$$\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{GMm}{\sqrt{(x+aU)^2+y^2+z^2}}$$

$$- \frac{GMm}{\sqrt{(x-aU)^2+y^2+z^2}} - \frac{m\omega^2}{2}(x^2+y^2) = h \quad (1.1)$$

where $M$ and $M_p$ are masses of the Sun and the planet $P$ respectively, $a = SP$, the origin $O$ is the barycenter of the Sun and the planet $P$, the $X$-axis passes through $S$ and $P$, $U = M_p/(M+M_p)$, $h$ is a constant of integration, and $\omega$ is the angular velocity of $P$ about $S$. Let

$$x = aX, \ y = aY, \ \text{and} \ z = aZ$$

Dividing (1.1) by $ma^2\omega^2$, and applying Kepler's third law

$$a^3\omega^2 = G(M+M_p) \quad (1.2)$$

we have

$$\frac{X^2 + Y^2 + Z^2}{2\omega^2} = \frac{1-U}{\sqrt{(X+U)^2+Y^2+Z^2}}$$

$$- \frac{U}{\sqrt{(X-1+U)^2+Y^2+Z^2}} - \frac{X^2+Y^2}{2} = H \quad (1.3)$$
where

\[ H = h/ma^2\omega^2 \]

Let

\[ V = -\frac{1-U}{\sqrt{(X+U)^2+Y^2+Z^2}} - \frac{U}{\sqrt{(X-1+U)^2+Y^2+Z^2}} - \frac{X^2+Y^2}{2} \quad (1.4) \]

\[ T_r = (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)/2\omega^2 \quad (1.5) \]

The physical meaning of \( V \) is the function of potential energy including the eccentric potential which is expressed by the last term of (1.4). \( T_r \) is the kinetic energy related to rotating coordinate system \( O-XYZ \). Then (1.3) can be written as the form of energy integral:

\[ T_r + V = H \quad (1.6) \]

From (1.4), there are two singular points in \( V \), \((-U, 0, 0)\) and \((1-U, 0, 0)\). But they really do not appear. Because the sun and the planet have a finite volume, the particle cannot penetrate into them.

The value of \( V \) on the \( X \)-axis can be found by

\[ V_x = -\frac{1-U}{|X+U|} - \frac{U}{|X-1+U|} - \frac{X^2}{2} \quad (1.7) \]

In the solar system, the Jupiter is the most massive of the planets. Its \( U \) is \( U = 9.53864 \times 10^{-4} \). The value of \( U \) of the other planets are much smaller. The curve of \((X, V_x)\) is shown in Fig. 1. Let

\[ \frac{dV_x}{dx} = -\frac{1-U}{(x+U)^2} + \frac{U}{(X-1+U)^2} - X = 0 \quad (1.8) \]

Solving the above equation, we get three maximum points of \( V_x \): \( L_1 \), \( L_2 \), and \( L_3 \). They are equilibrium points of gravitational force with eccentric force. They are also called Lagrange points. Their \( X \)-coordinates may be found by the following expressions:

\[ X_1 = 1-U - (\alpha - \frac{\alpha^2}{3} - \frac{\alpha^3}{9} + \cdots) \quad (1.9) \]

\[ X_2 = 1-U + (\alpha + \frac{\alpha^2}{3} - \frac{\alpha^3}{9} + \cdots) \quad (1.10) \]

\[ X_3 = -1-U + \frac{7\alpha^3}{4+12\alpha^2} \quad (1.11) \]

where

\[ \alpha = [U/\{3(1-U)\}]^{1/3} \quad (1.12) \]

There are other two Lagrange points which lie on the \( OXY \) plane and each of them forms an equilateral-triangle configuration with the Sun \( S \) and the planet \( P \):