WEIGHTED MEDIAN ALGORITHMS FOR L_1 APPROXIMATION

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Abstract.

The weighted median problem arises as a subproblem in certain multivariate optimization problems, including L_1 approximation. Three algorithms for the weighted median problem are presented and the relationships between them are discussed. We report on computational experience with these algorithms and on their use in the context of multivariate L_1 approximation.

AMS Subject Classifications: 65D99, 65K10.

Keywords: L_1 approximation, median.

1. Introduction.

The weighted median problem is the following: given n unordered real numbers x_1, ..., x_n and associated positive real weights w_1, ..., w_n solve:

\[
\min_{x \in \mathbb{R}} f(x) = \min_{x \in \mathbb{R}} \sum_{i=1}^{n} w_i |x - x_i|.
\]

If the weights are all one, the problem reduces to that of finding a median of n numbers, for which a linear-time algorithm is well known (see [1], [6]). The left and right derivatives of f are respectively given by

\[
f_-'(x) = \sum_{i: x_i > x} w_i - \sum_{i: x_i \leq x} w_i
\]

\[
f_+'(x) = \sum_{i: x_i \geq x} w_i - \sum_{i: x_i < x} w_i
\]

Since f is convex and piecewise linear, a necessary and sufficient condition for x to solve (1) is:

\[
f_-'(x) \leq 0 \text{ and } f_+'(x) \geq 0,
\]
or equivalently,

\[ \text{wtsum}_1 \leq \text{wtsum}_2 + \text{wtsum}_3 \]
\[ \text{wtsum}_1 + \text{wtsum}_2 \geq \text{wtsum}_3 \]

where

\[ \text{wtsum}_1 = \sum_{i: x_i < x} w_i, \quad \text{wtsum}_2 = \sum_{i: x_i = x} w_i, \quad \text{and} \quad \text{wtsum}_3 = \sum_{i: x_i > x} w_i. \]

Clearly, a solution will always be given by one of the points \( x_i \), although there may in some cases be a continuum of solutions in the interval between two of the points.

If problem (1) is rewritten as

\[
\min_{x \in \mathbb{R}} \sum_{i=1}^{\infty} |w_i x - w_i x_i|
\]

and we let \( a_i = w_i, b_i = w_i x_i \), then problem (1) is equivalent to

\[
\min_{x \in \mathbb{R}} \sum_{i=1}^{n} |a_i x - b_i|
\]

which is the solution in the \( L_1 \)-norm to the overdetermined system of equations

\[
a x = b, \quad a, b \in \mathbb{R}^n.
\]

The problems arise also as a subproblem - namely the line search - in methods for solving the multidimensional linear \( L_1 \)-problem ([2], [3], [5]):

\[
\min_{x \in \mathbb{R}^m} \sum_{i=1}^{n} |a_i x - \beta_i|, \quad \text{where} \quad a_i \in \mathbb{R}^{m \times 1}, \quad \beta_i \in \mathbb{R}, \quad x \in \mathbb{R}^{1 \times m},
\]

and methods ([7], [14]) for minimizing a sum of Euclidean norms:

\[
\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \| A_i^T x - b_i \|_2, \quad \text{where} \quad A_i \in \mathbb{R}^{m \times l}, \quad b_i \in \mathbb{R}^l.
\]

Weighted medians can also be used as a basic technique in algorithms for nonlinear \( L_1 \)-approximation.

These line searches are generally implemented by a partial sorting technique, using either heapsort or quicksort. Linear-time weighted median algorithms were discovered independently by ([4], [12], [15], [16]), but have not been used in the context of mathematical programming. The purpose of this paper is to describe these various techniques for computing the weighted median and the relationships between them, as well as to report on computational experience with weighted median algorithms used in mathematical programming.