THE SOLUTION OF A CRACK EMANATING FROM AN ARBITRARY HOLE
BY BOUNDARY COLLOCATION METHOD

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Abstract:
In this paper a group of stress functions has been proposed for the calculation of a crack emanating from a hole with different shape (including circular, elliptical, rectangular, or rhombic hole) by boundary collocation method. The calculation results show that they coincide very well with the existing solutions by other methods for a circular or elliptical hole with a crack in an infinite plate. At the same time, a series of results for different holes in a finite plate has also been obtained in this paper. The proposed functions and calculation procedure can be used for a plate of a crack emanating from an arbitrary hole.

I. Introduction
It has been demonstrated that crack originates most frequently from stress concentration areas developed around holes. On the other hand, the engineering practice of "stress relief", in which holes are drilled at the ends of a thin slot with the object of reducing the concentration of stress at the base. Thus, a great deal of research has been conducted to determine the stress field around cracks emanating from holes.

The analysis of a crack emanating from a hole is usually fulfilled by analytical or numerical approach. For example, the problem of a radial crack situated at the edge of a circular hole was considered by Bowie by using a conformal mapping technique. Tweed and Rooke studied this problem by using Mellin transform technique. Rubinstein and Sadegh considered a circular hole with an arbitrary positioned edge crack by using dislocation density distribution function. For an elliptical hole with a crack Berezhnitskii studied it by using the integral equation method. Generally, the deficiency of these methods is that they only deal with the infinite plate case. For different shape of holes, the stress functions and solution approach are usually different.

As a common numerical approach, finite element method can certainly be used to calculate stress intensity factor (SIF) of a crack at the hole. But because of complexity of the geometrical shape and existence of stress concentration at the crack tip, the mesh refinement is necessary at these areas. Therefore, more preparation and computation time are needed, but the accuracy is usually not satisfactory.

The boundary collocation method in calculation of stress intensity factor has demonstrated its effectiveness. For example, the SIF values for single edge tension notch, and three point bend fracture specimens were calculated by Gross et al. By using the stress functions proposed by Kobayashi et al. SIF of a series of problems, including central or slant crack and two cracks at a
hole, has been calculated satisfactorily\(^{7-9}\). This method is rather simple, but has good accuracy, and is suitable for finite plates especially.

In this paper a group of stress functions has been proposed for the calculation of a crack emanating from a hole with different shapes. The analysis can be used for references in engineering applications.

II. Complex Stress Functions

As shown by Muskhelishvili\(^{10}\), the representation of the inplane stresses by two analytical functions \(\phi(z)\) and \(\omega(z)\) leads to a general method of solving two-dimensional plane problems. The stresses and displacements are related to these analytical functions as indicated by

\[
\begin{align*}
\sigma_{xx} + \sigma_{yy} &= 4\text{Re}\left[\Phi(z)\right] \\
\sigma_{xy} - i\tau_{xy} &= \Phi(z) + \Omega(z) + (z - z_0) \Phi'(z) \\
2G(u + iv) &= \kappa \phi(z) - \omega(z) - (z - z_0) \Phi(z)
\end{align*}
\]

where

\[
\begin{align*}
\Phi(z) &= \phi'(z) \\
\Omega(z) &= \omega'(z) \\
\kappa &= \begin{cases} 
3 - \nu & \text{plane stress} \\
3 - 4\nu & \text{plane strain}
\end{cases} \\
G &= \frac{E}{2(1+\nu)}
\end{align*}
\]

For the multiply connected regions the single-value displacement relation should be satisfied as

\[
\kappa \oint \Phi(z)\,dz - \oint \Omega(z)\,dz = 0
\]

The boundary conditions can be described by different approaches. In this paper the resultant force expression is used as

\[
\Phi(z) + \omega(\bar{z}) + (z - z_0) \Phi(\bar{z}) = -f_x + if_y + C_j \quad \text{on } L_j
\]

where \(C_j\) are certain constants for a contour \(L_j\). The resultant force components \(f_x\) and \(f_y\) are determined by the following integral:

\[
f_x - if_y = \int_{L_j} (X_n + iY_n)\,ds \quad \text{on } L_j
\]

in which \(z_\sigma\) denotes an arbitrarily fixed point on \(L_j\); \(X_n\) and \(Y_n\) are the given values of the components of the external loading at point \(z\).

Consider the plate with a crack at the hole under arbitrary loading shown in Fig. 1. For simplicity, the surfaces of the hole and crack have no loading. According to the characterization of the stress singularity at the crack tip and the internal loading-free surfaces, the complex stress functions are given in the following forms

\[
\begin{align*}
\Phi(z) &= G(z)F_1(z) + F_2(z) \\
\omega(z) &= G(z)F_1(z) - F_2(z)
\end{align*}
\]