EXACT ANALYTIC METHOD FOR SOLVING VARIABLE COEFFICIENT DIFFERENTIAL EQUATION

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Abstract

Many engineering problems can be reduced to the solution of a variable coefficient differential equation. In this paper, the exact analytic method is suggested to solve variable coefficient differential equations under arbitrary boundary condition. By this method, the general computation format is obtained. Its convergence is proved. We can get analytic expressions which converge to exact solution and its higher order derivatives uniformly. Four numerical examples are given, which indicate that satisfactory results can be obtained by this method.

I. Introduction

In engineering mechanics and other subjects, many practical problems can be reduced to the solution of a variable coefficient differential equation. For example, we solve the bending problems of nonhomogeneous beams and cylindrical shells. Hence, it is important to find a better method for solving the variable coefficient differential equation.

As is well-known, the exact solutions of variable coefficient differential equations obtained by the analytic method are only in some special cases. In general complicated cases, it is difficult to obtain the exact solution. The difference method is a classical method. By this method, the system of difference algebraic equations is substituted for variable coefficient differential equation. Because difference method is considered from global analysis, it is difficult to deal with complicated structure, boundary condition and loading in engineering problems. It can not design a standard program to solve the different kinds of boundary conditions. The finite element method is based on elements to solve the differential equation, hence it avoids the defects in difference method. But, it is a necessary condition that the differential operator should be positively definite.

The step reduction method is first suggested in [1] to solve nonhomogeneous elastic mechanics. The principle of step reduction method is that a nonhomogeneous body is divided into many elements. Every element can be regarded as homogeneous. Then, by continuity condition of physics, an analytical solution is obtained. It is considered from the view of physics, hence the step reduction method is limited in arbitrary variable coefficient differential equation.

On the basis of step reduction method, the exact analytic method is suggested to solve arbitrary variable coefficient differential equation. The method have many advantages as follows: Based on elements, it can solve the variable coefficient differential equation under complicated boundary and
loading. The non-positive definite differential operator can be solved and an analytic expression of solution can be given by this method. In engineering mechanics, the displacements and stress resultants have the same rate of convergence. The error analysis of numerical example indicates that the solution and its higher order derivatives obtained by the exact analytic method have second order rate of convergence. Solving \( k \)-th order differential equation, the problem can become the solution of the system of linear algebraic equations with \( k \) unknowns at most.

In this paper, the computation format to solve the variable coefficient differential equation and the general method to deal with arbitrary boundary conditions are given. Analytic expressions of solution are obtained, which converge to exact solution and its higher order derivatives uniformly. The convergence has been proved. Four numerical examples are given in this paper, which indicate that satisfactory results can be obtained by this method.

II. Theoretical Background of Exact Analytic Method

An arbitrary variable coefficient differential equation can be expressed as

\[
Aw(x) = \sum_{m=0}^{k} \left( P_m(x) w(x)^{\left(\nu_m\right)}\right)^{(u_m)} = f(x) \quad x \in [x_0, x_N]
\]  

(2.1)

where \( u_m \) and \( \nu_m \) are positive integers. We divide the interval \([x_0, x_N]\) into \( N \) elements. Suppose the interval of \( i \)-th element is \([x_{i-1}, x_i]\) and \( P_m(x) \in C^{\infty}(x_{i-1}, x_i) \) as well as \( f(x) \in L_2[x_0, x_N] \) in (2.1). Besides, we assume that the inverse operator \( A^{-1} \) exists under arbitrary \( f \in L_2[x_0, x_N] \) and given boundary conditions.

By the exact analytic method, in \( i \)-th element, equation (2.1) can be converted into

\[
\bar{A}_i\bar{w}(x) = \sum_{m=0}^{k} \left( P_m(x_i) \bar{w}(x)^{\left(\nu_m\right)}\right)^{(u_m)} = f(x) \quad x \in [x_{i-1}, x_i]
\]  

(2.2)

where \( \bar{x}_i = (x_{i-1} + x_i)/2 \). Let us define

\[
u = \max (\nu_m), \quad u = \max (u_m) \quad (m = 0, 1, \ldots, k)
\]  

(2.3)

and

\[
\bar{F}_j(x) = \sum_{m=0}^{k} \{u_m - j\}^* \left( P_m(x) \bar{w}(x)^{\left(\nu_m\right)}\right)^{(u_m - j)} \quad (j = u, u-1, \ldots, 1)
\]

\[
\bar{F}_j(x) = \sum_{m=0}^{k} \{u_m - j\}^* P_m(x_i) \bar{w}(x)^{\left(\nu_m + u_m - j\right)} \quad x \in [x_{i-1}, x_i]
\]

(2.4)

where

\[
\{u_m - j\}^* = \begin{cases} 1 & u_m \geq j \\ 0 & u_m < j \end{cases}
\]  

(2.5)