DIAGONALIZATION METHOD FOR A SINGULARLY PERTURBED VECTOR ROBIN PROBLEM

Ni Shou-ping (倪守平)

(Department of Mathematics, Fujian Normal University, Fuzhou)

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Abstract

In this paper the method and technique of the diagonalization are employed to transform a vector second-order nonlinear system into two first-order approximate diagonalized systems. The existence and the asymptotic behavior of the solutions are obtained for a vector second-order nonlinear Robin problem of singular perturbation type.

I. Introduction

We consider here the existence and the asymptotic behavior as \( \varepsilon \to 0^+ \) of solutions of the singularly perturbed boundary value problem

\[
\begin{align*}
\varepsilon y'' &= f(t, \varepsilon, y, y') & 0 < t < 1 \\
A_1(\varepsilon)y(0, \varepsilon) + A_2(\varepsilon)y'(0, \varepsilon) &= \alpha(\varepsilon) \\
B_1(\varepsilon)y(1, \varepsilon) + B_2(\varepsilon)y'(1, \varepsilon) &= \beta(\varepsilon)
\end{align*}
\]

where \( y, f, \alpha, \beta \) are \( n \)-dimensional vector functions and \( A_i, B_i, i=1,2 \) are \( n \times n \) matrix functions and \( |\det A_1(\varepsilon)| + |\det A_2(\varepsilon)| > 0, \quad |\det B_1(\varepsilon)| + |\det B_2(\varepsilon)| > 0 \).

For vector boundary problems of singular perturbation type, diagonalization method was given by K.W.Chang\(^1\) who employed these results to consider the singular perturbation for a vector second-order nonlinear Dirichlet problem\(^2\),\(^3\). In [4] we extended the diagonalization method to higher order differential equation, and thus obtained the existence and the asymptotic behavior of the solutions for a class of vector higher-order nonlinear boundary value problems with boundary perturbations. These results developed the application of diagonalization method.

A natural question is: Can we employ the diagonalization method to treat the vector nonlinear Robin problems? It was proposed by Professor K.W.Chang in the lectures. However, this question does not appear to have been studied till now. In this paper we shall improve the method of the demonstration referred to in the above papers, and so solve the question. Our main results for vector Robin problem are analogous to those obtained by Bris\(^5\) for the scalar case.

II. Preliminary Result

We need the following basic result concerning diagonalization method (see [1] and [2]).

Consider the vector linear differential equation

\[
\varepsilon v\cdot = C(t, \varepsilon)v + D(t, \varepsilon)v' + g(t).
\]

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where \( C(t, \varepsilon) = C(t, 0) + \mathcal{O}(\varepsilon) \) and \( D(t, \varepsilon) = D(t, 0) + \mathcal{O}(\varepsilon) \) are continuous and bounded \( n \times n \) matrix functions for \( 0 \leq t \leq 1 \) and \( \varepsilon > 0 \).

**Lemma** Let every eigenvalue of \( D(t, 0) \) have a real part \( \leq -8\mu < 0 \) for \( 0 \leq t \leq 1 \). Then there exists \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( 0 \leq t \leq 1 \), the following matrix initial value problems

\[
e_p' = D(t, \varepsilon)P + \varepsilon P^2 - C(t, \varepsilon), \quad P(0, \varepsilon) = 0
\]

\[
\varepsilon Q' = -eP(t, \varepsilon)Q - Q[D(t, \varepsilon) + \varepsilon P(t, \varepsilon)] - I, \quad Q(1, \varepsilon) = 0
\]

have respectively solutions \( P(t) = P(t, \varepsilon) \) and \( Q(t) = Q(t, \varepsilon) \) which are uniformly bounded, that is there exist positive constants \( p \) and \( q \) such that

\[
\|P(t)\| \leq p, \quad \|Q(t)\| \leq q
\]  
(2.2)

Moreover

\[
P(t, 0) = \lim_{\varepsilon \to 0^+} P(t, \varepsilon) = D^{-1}(t, 0)C(t, 0), \quad 0 \leq t \leq 1
\]

\[
Q(t, 0) = \lim_{\varepsilon \to 0^+} Q(t, \varepsilon) = -D^{-1}(t, 0), \quad 0 \leq t \leq 1
\]  
(2.3)

Furthermore, the change of variables

\[
z = v' + P(t, \varepsilon)v, \quad w = v + \varepsilon Q(t, \varepsilon)z
\]  
(2.4)

transforms (2.1) into the diagonalized system

\[
w' = -P(t, \varepsilon)w + Q(t, \varepsilon)g(t)
\]

\[
\varepsilon z' = [D(t, \varepsilon) + \varepsilon P(t, \varepsilon)]z + g(t)
\]  
(2.5)

Let \( W(t) = W(t, \varepsilon) \) with \( W(0, \varepsilon) = I \) be the fundamental matrix of the linear equation

\[
w' = -P(t, \varepsilon)w
\]

and \( Z(t) = Z(t, \varepsilon) \) with \( Z(0, \varepsilon) = I \) be the fundamental matrix of the linear equation

\[
\varepsilon z' = [D(t, \varepsilon) + \varepsilon P(t, \varepsilon)]z
\]

Then there exist \( \varepsilon_1 > 0 \) and \( L > 1 \) such that for \( 0 < \varepsilon \leq \varepsilon_1 \),

\[
|W(t)W^{-1}(s)| \leq L, \quad 0 \leq t, s \leq 1
\]

\[
|Z(t)Z^{-1}(s)| \leq L \exp \left[ -\mu(t-s)/\varepsilon \right], \quad 0 \leq s \leq t \leq 1
\]  
(2.6)

Furthermore, the general solution \( w(t) = w(t, \varepsilon), \ z(t) = z(t, \varepsilon) \) of (2.5) is

\[
w(t) = W(t)l_1 + \int_0^t W(t)W^{-1}(s)Q(s)g(s)ds
\]

\[
z(t) = Z(t)l_2 + \varepsilon^{-1} \int_0^t Z(t)Z^{-1}(s)g(s)ds
\]  
(2.7)

where \( l_i, i = 1, 2 \) are the arbitrary constant vectors.

**III. Boundary Layer Behavior**

We first consider the following Robin problem