BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

Zhou Qin-de (周钦德) Miao Shu-mei (苗树梅)

(Jilin University, Changchun)

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Abstract

In this paper the existence and uniqueness of solutions of boundary value problems

\[ \varepsilon y'' = f(t, y, \varepsilon) \]

\[ L(y(0), y'(0), \varepsilon) = 0, R(y(1), y'(1), \varepsilon) = 0 \]

(which contains the Robin's problem) is discussed by using the upper and lower solution. In addition, the asymptotic estimation of the solution is given as well.

I. Introduction

This paper studies the boundary value problems

\[ \varepsilon y'' = f(t, y, \varepsilon) \] (1.1)

\[ L(y(0), y'(0), \varepsilon) = 0, R(y(1), y'(1), \varepsilon) = 0 \] (1.2)

which involves a small parameter \( \varepsilon > 0 \). Previous work discussed the cases that the boundary condition is either \( y(0) = a, y(1) = b \) or \( y(0) - py'(0) = a, y(1) + qy'(1) = b, p \neq 0 \) (e.g. [1,2]). Our work contains the general Robin's boundary value problem and, in particular, discusses the uniqueness of solutions.

Let

\[ D = \{(t, y, \varepsilon) : 0 \leq t \leq 1, |y| < \infty, 0 \leq \varepsilon \leq \varepsilon_0 \} \]

where \( \varepsilon_0 \) is a positive constant. Suppose that

1. \( f(t, y, \varepsilon) \) and its partial derivatives with respect to \( y \) and \( \varepsilon \), respectively, are all continuous on \( D \).

2. There exists a positive number \( m \) and a function \( u(t) \), whose second order derivative is bounded, such that

\[ f_y(t, y, \varepsilon) \leq m, (t, y, \varepsilon) \in D ; \quad f(t, u(t), 0) = 0 \quad (0 \leq t \leq 1) \]

3. For each given \( \varepsilon \in [0, \varepsilon_0] \), both \( L(u, v, \varepsilon) \) and \( R(u, v, \varepsilon) \) are continuous functions on \(-\infty < u, v < \infty\), in addition, they are monotonic nondecreasing in \( v \).

We will quote Theorem 3.6 in [3], i.e.

Lemma 1 Assume that conditions 1 and 3 hold. If for each given \( \varepsilon \in (0, \varepsilon_0] \), there exist functions \( \bar{\omega}(t) \) and \( \underline{\omega}(t) \), such that

\[ \varepsilon \bar{\omega}''(t) \leq f(t, \bar{\omega}(t), \varepsilon), \quad \varepsilon \underline{\omega}''(t) \geq f(t, \underline{\omega}(t), \varepsilon) \] (1.3)
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\[
\begin{align*}
L(\bar{\alpha}(0), \bar{\alpha}'(0), \epsilon) &\leq 0, \quad R(\bar{\alpha}(1), \bar{\alpha}'(1), \epsilon) \geq 0 \quad (1.4) \\
L(\bar{\alpha}(0), \bar{\alpha}'(0), \epsilon) &\geq 0, \quad R(\bar{\alpha}(1), \bar{\alpha}'(1), \epsilon) \leq 0 \quad (1.5)
\end{align*}
\]

hold on \(0 \leq t \leq 1\), then the boundary value problem \((1.1), (1.2)\) has a solution \(y(t)\), satisfying the inequality

\[
\bar{\alpha}(t) \leq y(t) \leq \bar{\alpha}(t) \quad (0 \leq t \leq 1)
\]

II. Existence

In order to guarantee that the solution of boundary value problem exists, taking the general Robin’s problem as a background, we will restrict \(L(u, v, \epsilon)\) and \(R(u, v, \epsilon)\) further.

4. For any positive constant \(l\), there is a positive constant \(N\), such that

\[
L(u, -N, \epsilon) \leq 0, \quad L(u, N, \epsilon) \geq 0,
\]

as \(|u| \leq l\), \(0 \leq \epsilon \leq \epsilon_0\).

5. There is a positive constant \(N\) and small intervals \(u_1 < u < \bar{u}_1, \quad u_2 < u < \bar{u}_2 (\bar{u}_1 < 0 < u_2, \bar{u}_1 < u(0) < u_2)\), such that

\[
\begin{align*}
L(u, -N, \epsilon) &\leq 0, \quad (u_2 < u < \bar{u}_2, \quad 0 \leq \epsilon \leq \epsilon_0), \\
L(u, N, \epsilon) &\geq 0, \quad (u_1 < u < \bar{u}_1, \quad 0 \leq \epsilon \leq \epsilon_0).
\end{align*}
\]

4. For each positive constant \(l\), there is a positive constant \(N\), such that

\[
R(u, -N, \epsilon) \leq 0, \quad R(u, N, \epsilon) \geq 0,
\]

as \(|u| \leq l\), \(0 \leq \epsilon \leq \epsilon_0\).

5. There is a positive constant \(N\) and small intervals \(u_3 < u < \bar{u}_3, \quad u_4 < u < \bar{u}_4 (\bar{u}_3 < 0 < u_4, \bar{u}_3 < u(1) < u_4)\), such that

\[
\begin{align*}
R(u, -N, \epsilon) &\leq 0; \quad (u_3 < u < \bar{u}_3, \quad 0 \leq \epsilon \leq \epsilon_0), \\
R(u, N, \epsilon) &\geq 0; \quad (u_4 < u < \bar{u}_4, \quad 0 \leq \epsilon \leq \epsilon_0).
\end{align*}
\]

Theorem 1 Assume that conditions 1 - 3, 4, 4, hold. Then for \(\epsilon > 0\) sufficiently small, the boundary value problem \((1.1), (1.2)\) has a solution \(y(t, \epsilon)\) satisfying the inequality

\[
|y(t, \epsilon) - u(t)| \leq c_0 e^{q} \exp \left[ -\sqrt{\frac{m}{\epsilon}} t \right] + c_1 e^{q} \exp \left[ \sqrt{\frac{m}{\epsilon}} (t-1) \right] + \frac{M}{m} \epsilon \quad (2.1)
\]

on \(0 \leq t \leq 1\), where \(M\) is an upper bound of \(|f_r(t, u(t), \epsilon) - u''(t)|\) on \(0 \leq t \leq 1, \quad 0 \leq \epsilon \leq \epsilon_0\), \(c_1 (i = 0, 1)\) are positive constants satisfying \(\sqrt{m} c_1 \geq N; \quad |u'(i)| + 1; \quad N; (i = 0, 1)\) are the positive constants in condition 4 with \(l\) replaced by \(|u(t)| + 1\).

Proof Denote the function on the right-hand side of inequality \((2.1)\) by \(\omega(t, \epsilon)\), and let

\[
\bar{\omega}(t) = u(t) + \omega(t, \epsilon), \quad \omega(t) = u(t) - \omega(t, \epsilon).
\]

Then we can verify that they satisfy inequalities \((1.3) - (1.5)\) in Lemma 1. Since \(\bar{\omega}(t) - u(t) > 0\), so by the mean value theorem we know that there exist \(\xi \) between \(\bar{\omega}(t)\) and \(u(t)\), and \(\theta \in (0, 1)\) such that

\[
\begin{align*}
f(t, \bar{\omega}(t), \epsilon) &= f(t, \bar{\omega}(t), \epsilon) - f(t, u(t), \epsilon) + f(t, u(t), \epsilon) - f(t, u(t), 0) \\
&= f_r(t, \xi, \epsilon) ((\bar{\omega}(t) - u(t)) + [f_r(t, u(t), \theta), u''(t)] \epsilon + e u''(t) \\
&\geq m(\bar{\omega}(t) - u(t)) - M \epsilon + e u''(t).
\end{align*}
\]