ON AN INTERPOLATION-THEORETIC EXTREMUM PROBLEM

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To the memory of Professor PÁL TURÁN

In [5], p. 92, A. H. TURECKI mentioned the following unsolved problem: Let

\[ 0 \leq x_0 < x_1 < \ldots < x_{2n} < 2\pi, \]

and \( t_k(x) \) (\( k = 0, 1, \ldots, 2n \)) that uniquely determined trigonometric polynomial of degree \( n \) for which \( t_k(x_j) = \delta_{k,j} \) (\( k, j = 0, 1, \ldots, 2n \)). For what system of nodes (1) will

\[ I_p = I_p(x_0, \ldots, x_{2n}) = \int_0^{2\pi} \left| t_k(x) \right|^p dx \quad (0 < p < \infty) \]

be minimal? By a lengthy calculation, Turecki showed that in case \( p = 1 \) or \( p = 2 \), the necessary conditions \( \partial I_p / \partial x_k = 0 \) (\( k = 0, \ldots, 2n \)) of the minimum hold for the equidistant nodes

\[ t_k = t_{k,n} = \frac{2k\pi}{2n + 1} \quad (k = 0, \ldots, 2n). \]

However, the problem itself remained unsettled. Recently, R. SCHUMACHER [3] gave a solution in case \( p = 1 \), as a corollary of a much more general theorem. His method does not seem to be applied to the cases \( p \neq 1 \), because an essential use is made of the fact that \( I_1 \) is the integral of the Lebesgue function of the trigonometric interpolation.

In this note we give a simple and direct solution for the problem in the case \( 1 \leq p < \infty \). Perhaps our method can be applied to the case \( 0 < p < 1 \) as well, but at present I am unable to extend the proof in this direction. We also have some connected results for the algebraic case.


Key words and phrases. Linear operators, Lebesgue-functions and fundamental polynomials of interpolation.
THEOREM 1. Let \( p(\geq 1) \) be an arbitrary real number. The integral (2) is minimal if and only if the nodes (1) are identical with the equidistant nodes (3), or with their translation.

PROOF. We shall make use of the following result of MARCINKIEWICZ, ZYGMUND and BERMAN (see e.g. [4], pp. 481–481):

Let \( L_n \) be an arbitrary linear trigonometric polynomial operator which reproduces trigonometric polynomials of degree at most \( n \). Then

\[
\frac{1}{2\pi} \int_0^{2\pi} L_n(f(t-h), x+h) dh = s_n(f, x) \quad (-\infty < x < \infty)
\]

for all \( 2\pi \)-periodic Lebesgue-integrable function \( f(x) \), where

\[
s_n(f, x) = \frac{1}{n} \int_0^{2\pi} f(t+n)D_n(t)dt \quad \left( D_n(t) = \frac{\sin \frac{2n+1}{2}}{2 \sin \frac{t}{2}} \right)
\]

is the \( n^{th} \) Fourier partial sum of \( f(x) \).

Let especially

\[
L_n(f, x) = \sum_{k=0}^{2n} f(x_k) t_k(x)
\]

be the trigonometric interpolation polynomial based on the nodes (1). Then for \( x = 0 \), (4) takes the form

\[
s_n(f, 0) = \frac{1}{n} \int_0^{2\pi} f(t)D_n(t)dt = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{2n} f(x_k - h)t_k(h)dh.
\]

Let

\[
f(t) = f_1(t) = |D(t)|^{p-1} \text{sign } D(t),
\]

then

\[
s_n(f_1, 0) = \frac{1}{n} \int_0^{2\pi} f_1(t)D_n(t)dt = \frac{1}{2\pi} \int_0^{2\pi} |D_n(t)|^p dt
\]

and

\[
\frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{2n} f_1(x_k - h)t_k(h)dh = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{2n} |D_n(x - h)|^{p-1} \text{sign } D_n(x_k - h)dh.
\]

\[1\] For Lebesgue integrable functions \( f(x) \) this operator may be not well-defined. However, the relation (4) still holds in this case.