LAGRANGE EQUATION OF A CLASS OF NONHOLONOMIC SYSTEMS

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Abstract
Making use of conclusions from [1]: (1) $d-\delta$ operations are commutative; (2) the Appell-Chetaev condition restricting virtual displacements is superfluous, the present paper derives the Lagrange equation without multipliers for a class of first-order nonlinear nonholonomic dynamical systems by means of variational principle. This kind of equations is new.

Key words nonholonomic dynamics, $d-\delta$ commutativity, Appell-Chetaev condition

There used to be the opinion: (1) The commutativity of the operations—differentiation $d$ and variation $\delta$—is just a question of viewpoint; (2) The Appell-Chetaev condition is "a great event in the development of nonholonomic dynamics; which constitutes the foundation stone of nonlinear nonholonomic dynamics"[2,3]. We showed in [1]: (1) For first-order nonlinear nonholonomic systems $d-\delta$ operations are commutative; (2) The Appell-Chetaev condition is an unnecessary extra condition super-imposed on virtual displacements. In [1], according to these two conclusions, making use of the commutativity of $d-\delta$ operations and abandoning the Appell-Chetaev condition, we derived the Lagrange equation with multipliers. This equation has the same form as Vacco dynamics.

The Chaplygin equation[2] is the current equation of motion without multipliers for first-order nonlinear nonholonomic systems. Its form is complicated and the derivation is based on the traditional $d-\delta$ commutativity and the Appell-Chetaev condition.

Adopting the two above-mentioned conclusions and abandoning the traditional approach, the present paper, as a continuation of [1], derives the Lagrange equation without multipliers for a class of first-order nonlinear nonholonomic dynamical systems. This kind of equations is new and essentially differs from the current ones.

Let $q_1, \ldots, q_n$ be generalized coordinates of the system. It has $k$ independent first-order nonlinear nonholonomic constraints. As shown in [1], these $k$ constraints may be expressed as

$$f_i(q_1, \ldots, q_n; \dot{q}_{i+1}, \ldots, \dot{q}_n) = 0 \quad (i = 1, \ldots, k)$$

A class of constraints satisfying

$$\Delta = \frac{D(f_1, \ldots, f_k)}{D(q_1, \ldots, q_k)} \neq 0 \quad (1)$$

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will be considered in this paper. In virtue of the implicit function theorem,
\[ q_i = \varphi_i(q_{k+1}, \ldots, q_n; q_{k+1}', \ldots, q_n') \quad (i = 1, \ldots, k) \]  
(2)
can be obtained from the constraint condition, and
\[
\frac{\partial \varphi_i}{\partial q_j} = -\frac{1}{\Delta} \frac{D(f_1, \ldots, f_i, \ldots, f_n)}{D(q_1, \ldots, q_j, \ldots, q_n)},
\]
\[
\frac{\partial \varphi_i}{\partial q_j} = \frac{1}{\Delta} \frac{D(f_1, \ldots, f_i, \ldots, f_n)}{D(q_1, \ldots, q_j, \ldots, q_n)},
\]
with \( i = 1, \ldots, k; \ j = k + 1, \ldots, n \). (2) implies that \( q_{k+1}, \ldots, q_n \) may be chosen as variationally independent variables of the system. Thus, we have
\[
\delta q_i = \sum_{j=k+1}^{n} \left( \frac{\partial \varphi_i}{\partial q_j} \delta q_j + \frac{\partial \varphi_i}{\partial q_j} \delta q_j \right) \quad (i = 1, \ldots, k)
\]  
(3)

Now, we use the variational principle to derive the equation of motion. The variational principle states that among all possible motions of the dynamical system from instance \( t_0 \) to instance \( t_1 \), the real motion extremalizes the Hamiltonian action \( \int_{t_0}^{t_1} L(q, \dot{q}, t) dt \), where \( L(q, \dot{q}, t) \) is the Lagrangean of the system. Thus, under the condition \( \delta q(t_0) = \delta q(t_1) = 0 \), making use of the commutativity of \( d \cdot \delta \) operations, we have
\[
\delta \int_{t_0}^{t_1} L(q, \dot{q}, t) dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial q_i} \delta q_i \right) dt
\]
\[
= \int_{t_0}^{t_1} \sum_{i=1}^{n} \left( \frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) dt
\]
\[
= -\int_{t_0}^{t_1} \sum_{i=1}^{n} \delta_i(L) \delta q_i dt
\]  
(4)
where
\[
\delta_i = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i}
\]  
(5)
is the Eulerian operator. Substituting (3) into (4) yields
\[
-\delta \int_{t_0}^{t_1} L(q, \dot{q}, t) dt
\]
\[
= \int_{t_0}^{t_1} \left\{ \sum_{i=1}^{k} \delta_i(L) \left[ \sum_{j=k+1}^{n} \left( \frac{\partial \varphi_i}{\partial q_j} \delta q_j + \frac{\partial \varphi_i}{\partial q_j} \delta q_j \right) \right] + \sum_{j=k+1}^{n} \delta_i(L) \delta q_j \right\} dt
\]
\[
= \int_{t_0}^{t_1} \sum_{j=k+1}^{n} \left[ \delta_i(L) + \sum_{i=1}^{k} \delta_i(L) \frac{\partial \varphi_i}{\partial q_j} \delta q_j + \sum_{i=1}^{k} \delta_i(L) \frac{\partial \varphi_i}{\partial q_j} \delta q_j \right] dt
\]
\[
= \int_{t_0}^{t_1} \sum_{j=k+1}^{n} \left\{ \delta_i(L) - \sum_{i=1}^{k} \left[ \frac{d}{dt} \left( \delta_i(L) \frac{\partial \varphi_i}{\partial q_j} \right) - \delta_i(L) \frac{\partial \varphi_i}{\partial q_j} \right] \right\} \delta q_j dt.
\]
According to the variational principle, \( \delta \int_{t_0}^{t_1} L(q, \dot{q}, t) dt = 0 \) should be fulfilled for any set of variations \( \delta q_{k+1}, \ldots, \delta q_n \) of the independent variables. Thus, we finally arrive at