ON THE NONLINEAR STABILITY BEHAVIOUR OF
DISTORTED PLANE COUETTE FLOW

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Abstract

This paper discusses the nonlinear stability behaviour of distorted plane Couette flow to 2-dimensional disturbances, and compares it with that of distorted plane Poiseuille flow. The results show that plane Couette flow is more unstable than plane Poiseuille flow to finite-amplitude disturbances.

Key words distorted plane Couette flow, distorted plane Poiseuille flow, nonlinear stability behaviour

I. Introduction

In the experimental observations, both plane Poiseuille flow and plane Couette flow become unstable when Reynolds number is about 1000, but in stability analysis, these two flows belong in different classifications, the instability of plane Poiseuille flow belongs in subcritical range, and plane Couette flow, as well as pipe Poiseuille flow, belongs in bifurcation from infinity\(^{[1],[4]}\). The failure to obtain the critical Reynolds number for the latter two kinds of flow brings much difficulties to nonlinear stability analysis.

In reference [1], the author suggested a hydrodynamic stability theory of distorted laminar flow, and presented a kind of distortion profiles reflecting the nonlinear interaction between various disturbances. By using such distortions, th paper discussed the stability problem of parallel shear flows. In linear stability analysis, the stability behaviour of plane Poiseuille flow, plane Couette flow and pipe Poiseuille flow are investigated by a uniform method, the results indicated that these flows are all linear unstable under the circumstances of background perturbations. The nonlinear stability behaviours of plane Poiseuille flow and pipe Poiseuille flow are also discussed, numerical calculations showed some differences between these two flows.

This paper discusses the nonlinear stability behaviour of plane Couette flow, and finds out that we can make some comparison between its nonlinear stability behaviour and that of plane Poiseuille flow in spite of many differences. In this paper the author discusses in detail such similarities and differences.

II. The Nonlinear Evolution Equation of Disturbance Amplitude

By using the artificial neutrality method suggested by Prof. Zhou Heng, the nonlinear evolution equation of disturbance amplitude can be obtained as follows:
\[
\frac{da}{dt} = (A_{11}a + A_{21}a^2 + A_{31}a^3 + A_{41}e^4 + A_{41}e^4) a \\
\quad + (A_{22}a^2 + A_{32}a^3 + A_{42}e^4) a^3 + A_{43}e^4 a^5
\]  
(2.1)

or

\[
\frac{d(ea)}{dt} = (A_{11}e + A_{21}e^2 + A_{31}e^3 + A_{41}e^4)(ea) \\
\quad + (A_{23} + A_{33}e + A_{43}e^3)(ea)^3 + A_{44}(ea)^5 \\
= C_1(ea) + C_2(ea)^3 + C_3(ea)^5 = f(ea)
\]  
(2.2)

where \(C_1, C_2, C_3\) are constants determined by numerical calculations, \(C_1\) is just the linear growth rate \(\alpha C_1\) of disturbance amplitude.

When \(d(ea)/dt = 0\), we obtain the critical value \((ea)_{cr}\). When the 5th order term of \((ea)\) is involved, \(f(ea) = 0\) becomes a quadratic algebraic equation of \((ea)^2\), so in general situation, it has two real roots (or two equivalent real roots), we do not consider the complex roots and negative real roots.

\(f(ea)\) may have two situations as in figure 1, the arrow indicates the direction of the development of the disturbance amplitude \((ea)\) with time.

**Fig. 1** Situation of \(f(ea)\)

When \(-\frac{d(f(ea))}{d(ea)} \bigg|_{ea_{cr}} > 0\), the critical amplitude obtained is threshold amplitude, above which the disturbance grows; when \(-\frac{d(f(ea))}{d(ea)} \bigg|_{ea_{cr}} < 0\), the critical amplitude obtained is equilibrium amplitude, the disturbance amplitude will approach this value. We denote threshold amplitude by "...".

Because

\[
\frac{df(ea)}{d(ea)} = C_1 + 3C_1(ea)^3 + 5C_3(ea)^4
\]  
(2.3)

When two unequal positive real roots are obtained, if \(C_3 > 0\), the larger root is threshold amplitude, the smaller root is equilibrium amplitude; if \(C_3 < 0\), the situation is opposite. When only one positive real root is obtained, we have to calculate (2.3).

### III. Nonlinear Development of Linear Neutral Disturbances

In reference [1], we changed the amplitude of distortion profiles to make the flow into subcritical or supercritical range, and see the variation of coefficients in the nonlinear evolution equation (2.3). Here we use this method to calculate plane Couette flow, and compare the results with those of plane Poiseuille flow under the same conditions.