Converse results on equiconvergence of interpolating polynomials

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1. Introduction. Among the many results which have appeared in the last few years on Walsh equiconvergence and its extensions, a recent paper [3] of J. Szabados is one of the most interesting, both for its simplicity and for its novelty. In order to describe this result, we consider the class of functions \( A_q \) analytic in \( |z| < q \) but not in \( |z| \leq q \). (The class \( A(\varrho) \), on the other hand will denote the class of functions analytic in \( |z| < \varrho \).) Let \( L_{n-1}(f; z) \) denote the Lagrange interpolant to \( f \) on the \( n \)th roots of unity and let

\[
P_{n-1,j}(f; z) = \sum_{k=0}^{n-1} a_k z^{k+j}, \quad j = 0, 1, 2, \ldots,
\]

where

\[
f(z) = \sum_{k=0}^{\infty} a_k z^{k}, \quad \text{and} \quad \lim_{n \to \infty} |a_n|^{1/n} = 1/q.
\]

We then have the following generalization of Walsh's Theorem:

**Theorem A [1].** If for any \( f \in A_q \) and for any integer \( l \geq 1 \) we set

\[
\Delta_{l,n-1}(f; z) := L_{n-1}(f; z) - \sum_{j=0}^{l-1} P_{n-1,j}(f; z),
\]

and if \( \varrho > 1 \), we have

\[
\lim_{n \to \infty} \Delta_{l,n-1}(f; z) = 0, \quad |z| < \varrho^{l+1},
\]

where the convergence is geometric and uniform on closed subsets of \( |z| < \varrho^{l+1} \).

For \( l = 1 \), this gives Walsh's theorem [5]. For further details on this and its ramifications, we refer to a survey paper by R. S. Varga [4]. In this context, J. Szabados proved

**Theorem B.** If \( f(z) \) is analytic in \( |z| < 1 \) and continuous in \( |z| \leq 1 \), and if moreover \( \{\Delta_{l,n-1}(f; z)\} \) is uniformly bounded on compacts in \( |z| < \varrho^{l+1} \), then \( f \) is analytic in \( |z| < \varrho \).

This interesting theorem may be considered as a sort of converse of Theorem A. An analogue of Theorem B for the Hermite case has recently been proved [2]. More

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precisely, let \( f \) be analytic in \( |z|<1 \) and let \( f, f', \ldots, f^{(r-1)} \) be continuous in \( |z|<1 \). If \( A_{r-1}(f; z) \) is uniformly bounded on compacts in \( |z|<q^{1+r/n} \), then \( f \) is analytic in \( |z|<q \). (For the definition of \( A_{r-1}(f; z) \) and further details see [1] and [2]).

The object of this note is to make the above Theorem B of Szabados more precise. Indeed we consider two entities — a null sequence \( \{a_k\}_0^\infty \) and a function \( f \) defined on all the roots of unity, i.e., on \( \bigcup_{n=1}^\infty U_n \), where \( U_n=\left\{ \exp \frac{2k\pi i}{n}, \ k=0, 1, \ldots, n-1 \right\} \).

For such a function \( f \), \( L_{n-1}(f; z) \) has a meaning and so we define \( A_{l,n-1}(f; z) \), depending on \( f \) and the null sequence \( \{a_k\}_0^\infty \), where \( A_{l,n-1}(f; z) \) is given explicitly by (2.1). We shall show that if we require \( A_{l,n-1}(f; z) \) to be uniformly bounded on compact subsets of \( |z|<q^{1+l} \), then the function \( f \) can be extended to be the sum of two functions \( g \) and \( h \), where \( g(z)=\sum_{k=0}^\infty a_k z^k \) is analytic in \( |z|<q \) and \( h \) is analytic in \( |z|<q^{1+l} \). Moreover, if we further require that \( A_{l,n-1}(f; z) \) tend to zero on an infinite set with a limit point in \( |z|<q^{1+l} \), then \( \sum_{k=0}^\infty a_k z^k \) is the analytic extension of \( f \) and is analytic in \( |z|<q \).

In Section 2, we state the main results. Section 3 deals with some lemmas needed later. Section 4 comprises the proof of Theorems 1 and 2. It would be interesting to find suitable analogues of Theorems 1 and 2 in the Hermite case and in the case of \( l_2 \)-approximation. A few remarks on the extension of Theorems 1 and 2 to the case of next-to-interpolatory polynomials or their iterations are added at the end of Section 4.

2. Statement of Results. Let \( U_n=\{z: z=\exp 2k\pi i/n, k=0, 1, \ldots, n-1 \} \) and let \( U=\bigcup_{n=1}^\infty U_n \) be the roots of unity and let \( \{a_k\}_0^\infty \) be a null sequence of real or complex numbers. For any function \( f \) defined on \( U \) and for any fixed integer \( l \) we define \( A_{l,n-1}(z) \) by (1.2), or equivalently

\[
A_{l,n-1}(z) := A_{l,n-1}(f; \{a_k\}_0^\infty; z) := L_{n-1}(f; z) - \sum_{k=0}^{n-1} \sum_{j=0}^{l-1} a_k z^j,
\]

where \( L_{n-1}(f; z) \) is the Lagrange interpolant to \( f \) on \( U_n \). If \( A(q) \) denotes the class of functions analytic in \( |z|<q \), we shall prove

**Theorem 1.** The sequence \( \{A_{l,n-1}(z)\}_{n=1}^\infty \) is uniformly bounded on compact subsets of \( |z|<q^{1+l} \) if and only if

(i) the function \( g(z):=\sum_{k=0}^\infty a_k z^k \in A(q) \) and

(ii) there exists a function \( h(z) \in A(q^{1+l}) \) such that

\[
(g+h)(z) = f(z), \quad z \in U.
\]