VARIANTS OF ROBINSON'S ESSENTIALLY UNDECIDABLE THEORY $R^*$

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Abstract

Cobham has observed that Raphael Robinson's well known essentially undecidable theory $R$ remains essentially undecidable if the fifth axiom scheme $(x \leq \bar{n} \lor \bar{n} \leq x)$ is omitted. We note that whether the resulting system is in a sense "minimal essentially undecidable" depends on what the basic constants are taken to be. We give an essentially undecidable theory based on three axiom schemes involving only multiplication and less than or equals.

1. Minimality of Cobham's Fragment of $R$

Raphael Robinson's well known essentially undecidable theory $R$ [4] is based on five axiom schemes:

\begin{align*}
\Omega_1 & : \bar{n} \cdot \bar{p} = n \cdot p \\
\Omega_2 & : \bar{n} + \bar{p} = n + p \\
\Omega_3 & : \bar{n} \cdot \bar{p} \quad \text{for} \quad n \neq p \\
\Omega_4 & : x \leq \bar{n} \rightarrow x = 0 \lor \ldots \lor x = \bar{n} \\
\Omega_5 & : x \leq \bar{n} \lor \bar{n} \leq x \\
\end{align*}

In the original version the constants are 0, $S$ (successor), + and ·. The symbol $\bar{n}$ denotes the $n$th numeral, defined recursively by

$$\bar{0} = 0, \quad \bar{n+1} = S\bar{n},$$

and $x \leq \beta$ stands for $(\exists w) (w + x = \beta)$ where $w$ is a variable occurring in neither $x$ nor $\beta$. It is still essentially undecidable if the constant $S$ is removed and replaced by constants $\bar{1}, \bar{2}, \bar{3}, \ldots$, and if $\leq$ is taken as a primitive constant instead of being defined in terms of +. Indeed the proof in [4] that every recursive function is

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definable in \( R \) needs only to be changed by replacing \( Su=v \) by \( u+1=v \) as a defining formula for the successor function.

Cobham (see [5]) observed that the theory \( R_0 \) based on the first four schemes is also essentially undecidable. Indeed \( R \) is interpretable in \( R_0 \) for if one defines

\[
Su=v \rightarrow \{[0 \leq y \land (\forall u)(u \leq y \land u \neq y \rightarrow u+1 \leq y)] \rightarrow x \leq y\},
\]

then \( \leq' \) satisfies \( \Omega_4 \) and \( \Omega_3 \) (indeed a further modification of \( \leq \) satisfies the stronger axiom \( x \leq y \lor y \leq x \)). The resulting axiom system \( \Omega_1, \Omega_2, \Omega_3, \Omega_4 \) is minimal essentially undecidable in the sense that if one omits any one of these entire schemes the resulting theory is easily seen to be not essentially undecidable. Namely to get a decidable complete extension of the system with \( \Omega_1 \) omitted use the theory of the reals with a non standard definition \( x+y=-1 \) for all \( x, y \); with \( \Omega_2 \) omitted use arithmetic with \( x \cdot y \) defined as \( x+y \); with \( \Omega_3 \) omitted use a one-element model; with \( \Omega_4 \) omitted use the reals. However one may omit infinitely many instances of each of the schemes and still leave an essentially undecidable theory. For example it's obviously enough to have only the instances with all numbers even; and, as was pointed out in [4], one only needs \( \Omega_2 \) for the cases \( n=p \).

If the constant \( S \) is replaced by constants \( 1, 2, 3, \ldots \) the theory obviously remains essentially undecidable. But if \( \leq \) is taken as a primitive predicate constant instead of being defined in terms of + then it is no longer essentially undecidable, a decidable complete extension being given by the theory of the reals, with \( x \leq y \) defined as always false. However if one strengthens \( \Omega_4 \) to \( \Omega_4' \):

\[
\Omega_4' \quad x \leq n \leftrightarrow x = 0 \lor \ldots \lor x = n,
\]

then one does obtain an essentially undecidable theory (call it \( R_0' \)), for the \( \leq' \) defined above then satisfies \( \Omega_4 \) and \( \Omega_3 \). So \( R \) is interpretable in \( R' \). Note that when \( \leq \) is defined as above in terms of +, then \( \Omega_4' \) follows from \( \Omega_4 \) and \( \Omega_1 \). However this theory, \( R_0' \), is not minimal essentially undecidable. We can delete \( \Omega_1 \).

**Theorem.** The theory \( R_1 \) based on the following three axiom schemes is essentially undecidable:

\[
\begin{align*}
\Omega_2 & \quad n \cdot \overline{p} = n \cdot \overline{p}, \\
\Omega_3 & \quad n \neq \overline{p} \quad \text{for} \quad n \neq p, \\
\Omega_4 & \quad x \leq n \leftrightarrow x = 0 \lor \ldots \lor x = n.
\end{align*}
\]

**Proof.** Here \( \leq' \) is to be considered a primitive undefined predicate constant and \( 0, 1, 2, \ldots \) are arbitrary constants. We will use Julia Robinson's definition [2] of addition in terms of multiplication and successor. Suppose that \( \Omega_2, \Omega_3, \Omega_4 \) are satisfied. Then we may define \( a+ \) which satisfies \( \Omega_1 \) as follows. First define the successor relation by

\[
y = x + 1 \leftrightarrow (x = 0 \land y = 1) \lor (x = 1 \land y = 2) \lor (x < y \leq x^2 \land \neg \exists z(x < z < y)),
\]