DEFINABLE RAMSEY AND DEFINABLE ERDÖS ORDINALS*

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Abstract

This paper studies the relation between definable Ramsey ordinals and constructible sets which have a certain set of indiscernibles. It is shown that an ordinal \( \kappa \) is \( \Sigma_1 \)-Ramsey if and only if \( \kappa \) is \( \Sigma_\omega \)-Ramsey. Similar results are obtained for definable Erdős ordinals.

1. Preliminaries

All the results in the present paper can be formulated in ZFC (Zermelo Fraenkel set theory with the axiom of choice).

If \( X, Y \) are two sets, then \( X \Delta Y \) is the symmetric difference of \( X \) and \( Y \). If \( f : X \to Y \) is a function and \( Z \subseteq X \), then \( f[Z] = \{ f(z) : z \in Z \} \).

If \( x = \langle x_1, \ldots, x_n \rangle \) is an \( n \)-tuple, then \( n \) is called the length of \( x \); \( (x)_i = x_i \) if \( 1 \leq i \leq n \) and \( = 0 \) otherwise; \( \langle \rangle = 0 \) is the empty sequence.

If \( < \) is a linear ordering on a set \( X \) and \( n < \omega \), then \( [X]^n = \{ \langle x_1, \ldots, x_n \rangle : x_1 < x_2 < \cdots < x_n \} \) and \( [X]^\omega = \bigcup [X]^n : n < \omega \). If \( \langle X, < \rangle \) is well-ordered, then \( \text{tp}(\langle X, < \rangle) \) is the unique ordinal order isomorphic to \( \langle X, < \rangle \); if \( < \) is easily understood, then \( \text{tp}(X) \) abbreviates \( \text{tp}(\langle X, < \rangle) \); \( \text{tp}(X) \) is called the order type of \( X \).

This paper is concerned with formulas and structures in the language \( \{ \in \} \). The symbol \( \varphi(v_0, \ldots, v_n) \) means that \( \varphi \) is a formula whose free variables form a subset of \( \{ v_0, \ldots, v_n \} \). Structures \( \langle M, \in \rangle, M \rangle \) of the language \( \{ \in \} \) are usually abbreviated by \( M \). \( \Sigma_n, \Pi_n \) formulas, where \( 0 \leq n \leq \omega \), are defined as usual. \( \models_M \Sigma_n \) is the satisfaction relation on \( M \) for \( \Sigma_n \) formulas. For structures \( M, N \) as above, \( M <_n N \) if \( M \subseteq N \) and both \( M, N \) satisfy the same \( \Sigma_n \) sentences with parameters in \( M \). Let \( N \subseteq M \). An \( m \)-ary relation \( R \subseteq N^m \) is said to be in \( \Sigma_n^M(M) \) [resp. \( \Pi_n^M(M) \)] if there exists a \( \Sigma_n \) (resp. \( \Pi_n \)) formula \( \varphi(v_1, \ldots, v_m, u_1, \ldots, u_q) \) and parameters \( a_1, \ldots, a_q \in M \) such that

\[
R = \{ \langle x_1, \ldots, x_m \rangle \in N^m : N \models \varphi(x_1, \ldots, x_m, a_1, \ldots, a_q) \}.
\]

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$\Delta^N_n(M) = \Sigma^N_n(M) \cap \Pi^N_n(M)$. $\Sigma^M_n(M)$, $\Sigma^M_n(\emptyset)$ are abbreviated by $\Sigma_n(M)$, $\Sigma^M_n$, respectively (similarly for $\Pi_n$, $\Delta_n$).

Definitions and properties of primitive recursive functions and relations can be found in [4] or [8]. The Jensen hierarchy $\langle J_\alpha : \alpha \in \text{Ord} \rangle$ of constructible sets is defined in [4]. The mapping $\alpha \mapsto J_\alpha$ is primitive recursive. A $\Sigma_1$-skolem function for $J_\alpha$ is a $\Sigma^1_1$ function $h$ whose domain is a subset of $\omega \times J_\alpha$ such that whenever $\forall x \in J_\alpha$, then $\exists y (y = h(x))$. The following theorems can be proved easily using fine structure results.

**Theorem 1.1.** There exists a $\Sigma_1$ formula $H$ without any parameters such that for any $\alpha > 0$ if $h_\alpha$ is defined by

$$h_\alpha(n, x) = y \text{ iff } J_\alpha \models H(n, y, x),$$

then $h_\alpha$ is a $\Sigma_1$ skolem function of $J_\alpha$.

**Theorem 1.2** ($\Sigma_1$-Selection Theorem). For any $\alpha > 0$, $p \in J_\alpha$ and any relation $R \subseteq \Sigma^1_1(p)$ on $J_\alpha$, there exists a $\Sigma^1_1(p)$ function $r$ such that

(i) $\text{dom}(r) = \text{dom}(R)$

(ii) $\forall x \in J_\alpha \exists y (y R(x, x) \leftrightarrow R(r(x), x)).$

**Theorem 1.3** (Condensation Lemma). If $M <_1 J_\alpha$, then there exists $\beta \leq \alpha$ such that $M \cong J_\beta$ (i.e. $\langle M, e \rangle$ is $\in$-isomorphic to $\langle J_\beta, e \rangle$).

**Theorem 1.4.** If $1 \leq n < \omega$, then $\models \Sigma^1_n$ is $\Sigma^1_n$ uniformly on $\alpha > 0$.

Also recall from [1] that KP is the Kripke-Platek set theory; let KP$_\omega$ be KP plus the axiom of infinity.

### 2. The Basic Definitions

Throughout we assume $1 \leq n \leq \omega$ and $\kappa = \omega \kappa > \omega$ is an ordinal.

**Definition 2.1.** Let $\langle M, e \upharpoonright M \rangle$ be an $e$-structure with universe $M$ and let $<$ be a linear ordering on a subset of $M$.

(a) Let $\Phi$ be a set of formulas in the language $\{e\}$. For each $I$ contained in the field of $<$, $I$ is called $\Phi$-$<$-homogeneous for $M$ or a set of $\Phi$-$<$-indiscernibles for $M$, if the following two conditions hold

(i) $I$ is infinite and

(ii) for any $m < \omega$, for any formula $\varphi(v_0 \ldots v_{m-1})$ in $\Phi$ and for any sequences $i_0, \ldots, i_{m-1}$, $j_0, \ldots, j_{m-1}$ of elements of $I$, if $i_0 < \ldots < i_{m-1}$ and $j_0 < \ldots < j_{m-1}$, then

$M \models \varphi(i_0, \ldots, i_{m-1}) \iff \varphi(j_0, \ldots, j_{m-1}).$

(b) Similarly, if $\Gamma$ is a class of relations on $M$ the notion of $I$ is a set of $\Gamma$-$<$-indiscernibles for $M$ is defined by

(i) $I$ is infinite and

(ii) for any $m < \omega$, for any $m$-ary relation $R$ in $\Gamma$ and for any sequences $i_0, \ldots, i_{m-1}$,