Efficient and Optimal Portfolios by Homogeneous Programming\(^1\)

By P. van Moeseke, Louvain\(^2\) and B. von Hohenbalken, Alberta\(^3\)

Eingegangen am 5. Oktober 1972
Revidierte Fassung eingegangen am 16. Juli 1973

Summary: Some general results of efficiency theory are applied to the selection of portfolios on the basis of the first two moments of their yield distributions. An arbitrary efficient portfolio can be computed by homogeneous programming. The corresponding duality theorem selects, given the market interest rate, an optimal portfolio from the efficient set.

The first three sections develop this specialized (and sharper) theory for the portfolio selection case. Section 4 applies it to a 54 stock example.


Die ersten drei Abschnitte enthalten die theoretischen Entwicklungen, Abschnitt 4 bringt ein Beispiel mit 54 Aktien.

1. Introduction

Suppose one characterizes a portfolio by the first two moments of the (investor's subjective) yield distribution, viz. expected profit and profit variance, where the latter is a risk measure. The portfolio is Markowitz [1959, ch. 7] efficient if no portfolio with at least equal profit expectation has a smaller profit variance and no portfolio with at most equal variance has a larger expected profit. (The algebraic formulation of this efficiency concept is (2.2) below).

It has been shown that one can always select an efficient portfolio by homogeneous [Moeseke, 1965] or by quadratic [Markowitz, op. cit] programming. (The relevant programming problems are stated in (2.4) and proposition 2.4 (4) below, respectively.)

However, the set of efficient decisions is usually broad. The crux of the problem is the additional criterion that selects an optimal portfolio among the efficient ones.

---

\(^1\) Earlier versions of this paper were presented at the Belgian-Israeli Colloquium on Operations Research, Technion, Haifa, June 1970, and at the Second World Congress of the Econometric Society, Cambridge, England, September 1970.

\(^2\) Prof. Paul van Moeseke, Université Catholique de Louvain, Louvain, Belgique.

\(^3\) Balder von Hohenbalken, Professor of Economics, Department of Economics, University of Alberta, Edmonton, Alberta, Canada T6G 2H4.
One readily observes that the minimax rule, as well as related rules (maximax, α-criterion, minimax regret) are not generally relevant here. Apart from the arbitrary risk attitude implied by such rules they may not yield a solution: if, say, the yield distributions have the entire real line as domain (as in the normal case) the minimum yield of every portfolio is $-\infty$ and the minimax is indeterminate (section 2).

In the last section we show that, given the market rate of interest, the duality theorem of homogeneous programming (see Moeseke [1965] and Markowitz [1959]) allows the selection of an optimal portfolio among the efficient ones.

2. Efficient Portfolios

In general, be $X \subset \mathbb{R}^n$ a set of feasible decisions, $f: X \to \mathbb{R}^m$ an $m$-tuple of real-valued functions $f_i, i = 1, \ldots, m$. Be $y, z \in \mathbb{R}^m$ and by $y \succeq z$ denote $y_i \geq z_i$ (all $i$); by $y \succeq z$ denote $y_i \geq z_i$ and $y \neq z$; by $y \succeq z$ denote $y_i > z_i$ (all $i$). Decision $x^* \in X$ is efficient relatively to $f$ if $f(x^*)$ is maximal in $f(X)$ for the partial order $\succeq$ on $\mathbb{R}^m$, i.e. if $X$ owns no $x$ such that $f(x) \succeq f(x^*)$.

Specifically, for the portfolio problem, denote the yield of a portfolio by

$$h(x) = cx, \quad x \in X; \quad X = \{x \geq 0 | q x < 1\}; \quad x,c,q \in \mathbb{R}^n, q > 0,$$

where $x$ is a portfolio containing $x_i$ units of the $i$th security; $c$, the $n$-tuple of yields (dividends plus net capital gains per accounting period); $q$ the $n$-tuple of security prices. Without loss of generality we fix the investor’s budget at unity.

Markowitz [ibid.] has suggested, for stochastic $c$ with (subjective) distribution $D$, first moments $\bar{c} = \int c dD$, and second moments $V = [\sigma_{ij}] = \int (c_i - \bar{c}_i)(c_j - \bar{c}_j) dD$, that the investor restrict his attention to portfolios $x^*$ that satisfy

$$\sigma h(x^*) \leq \sigma h(x), \quad \text{all } x \in X \quad \text{such that} \quad Eh(x) \geq Eh(x^*),$$

$$Eh(x^*) \geq Eh(x), \quad \text{all } x \in X \quad \text{such that} \quad \sigma h(x) \leq \sigma h(x^*),$$

where $Eh(x) = \bar{c} x$, $\sigma h(x) = (x V x)^{1/2}$. Putting $f_1(x) = Eh(x)$, $f_2(x) = -\sigma h(x)$, one sees readily that (2.2) is a special case of the general efficiency criterion cited above.

We need the following propositions. All propositions in this and the following sections have been lifted from Moeseke and v. Hohenbalken [1968], where the reader is referred for the proofs. The first proposition characterizes efficient decisions $x^*$ as maximizers on $X$ of the scalar function $\sum p_i f_i$, or $pf$ for short.

Theorem 2.1:

(a) If $p > 0$ every maximizer $x^*$ of $pf$ is efficient.

(b) If $f(X)$ is closed, bounded above4), and nonempty then $pf$ possesses an efficient maximizer $x^*$ for every $p > 0$.

4) With respect to the partial order $\succeq$ on $\mathbb{R}^n$. 