SOME GREEDY t-INTERSECTING FAMILIES OF FINITE SEQUENCES*

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Abstract

Let \( n, s_1, s_2, \ldots \) and \( s_n \) be positive integers. Assume \( \mathcal{M}(s_1, s_2, \ldots, s_n) = \{(x_1, x_2, \ldots, x_n) \mid 0 \leq x_i \leq s_i, \ x_i \ \text{is an integer for each } i\} \). For \( a = (a_1, a_2, \ldots, a_n) \in \mathcal{M}(s_1, s_2, \ldots, s_n) \), \( \mathcal{F} \subseteq \mathcal{M}(s_1, s_2, \ldots, s_n) \), and \( A \subseteq \{1,2,\ldots,n\} \), denote \( s_p(a) = \{j \mid 1 \leq j \leq n, \ a_j \geq p\} \), \( S_p(\mathcal{F}) = \{s_p(a) \mid a \in \mathcal{F}\} \), and \( W_p(A) = p^{n-|A|} \prod_{i \in A} (s_i - p) \).

\( \mathcal{F} \) is called an \( I_p \)-intersecting family if, for any \( a, b \in \mathcal{F} \), \( a_i \cap b_i = \min(a_i, b_i) \geq p \) for at least \( t \) \( i \)'s. \( \mathcal{F} \) is called a greedy \( I_p \)-intersecting family if \( \mathcal{F} \) is an \( I_p \)-intersecting family and \( W_p(\mathcal{A}) > W_p(\mathcal{B} + A) \) for any \( A \subseteq \mathcal{S}_p(\mathcal{F}) \) and any \( B \subseteq A \) with \( |B| = t + 1 \).

In this paper, we obtain a sharp upper bound of \( |\mathcal{F}| \) for greedy \( I_p \)-intersecting families in \( \mathcal{M}(s_1, s_2, \ldots, s_n) \) for the case \( 2p \leq s_i \ (1 \leq i \leq n) \) and \( s_1 > s_2 > \ldots > s_n \).

Key words. \( I_p \)-greedy subsets, \( I_p \)-regular subset, \( t \)-intersecting family, \( I_p \)-intersecting family, greedy \( I_p \)-intersecting family

1. Introduction

Let \( n, s_1, s_2, \ldots \) and \( s_n \) be positive integers. Assume \( \mathcal{M}(s_1, s_2, \ldots, s_n) = \{(x_1, x_2, \ldots, x_n) \mid 0 \leq x_i \leq s_i, \ x_i \ \text{is an integer for each } i\} \), which is equivalent to the poset of the product of \( n \) chains \( I_i \) and \( |I_i| = s_i + 1 \ (1 \leq i \leq n) \). Meanwhile, for any \( x = (x_1, x_2, \ldots, x_n) \in \mathcal{M}(s_1, s_2, \ldots, s_n) \), we can have a multiset \( A \) of \( X = \{1,2,\ldots,n\} \) with \( i \) appearing in \( A \) for \( x_i \) times. So \( \mathcal{M}(s_1, s_2, \ldots, s_n) \) can be regarded as the collection of all multisubsets of \( X \).

Throughout, \( X = \{1, 2, \ldots, n\} \) denotes a finite set. For an integer \( k \) with \( 0 \leq k \leq n \), \( C^n_k \) denotes a collection of all \( k \)-subsets of \( X \). For subset \( A, |A| \) denotes the number of the elements of \( A \). For \( a = (a_1, a_2, \ldots, a_n) \in \mathcal{M}(s_1, s_2, \ldots, s_n) \), the support of \( a \) is defined as \( S(a) = \{j \mid 1 \leq j \leq n, \ a_j > 0\} \). For \( \mathcal{F} \subseteq \mathcal{M}(s_1, s_2, \ldots, s_n) \), the support of \( \mathcal{F} \) is defined as \( S(\mathcal{F}) = \{s(a) \mid a \in \mathcal{F}\} \).

Furthermore, \( s_p(a) = \{j \mid 1 \leq j \leq n, \ a_j \geq p\} \) is called the \( p \)-support of \( a \), and \( S_p(\mathcal{F}) = \{s_p(a) \mid a \in \mathcal{F}\} \) is called the \( p \)-support of \( \mathcal{F} \). For a subset \( A \subseteq \{1, 2, \ldots, n\} \), \( W(A) = \prod_{i \in A} (s_i) \) is called the weight of \( A \) and \( W_p(A) = p^{n-|A|} \prod_{i \in A} (s_i - p) \) is called the \( p \)-weight of \( A \). Besides, we denote \( \{t\} = \{1, 2, \ldots, t\} \).


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Most of our other notation and terminology are the same as those in [1] and [10].

Definition 1. Let \( a = (a_1, a_2, \ldots, a_n) \), \( b = (b_1, b_2, \ldots, b_n) \) \( \in \mathcal{M}(s_1, s_2, \ldots, s_n) \). The intersection of \( a \) and \( b \) is defined as \( a \cap b = (a_1 \wedge b_1, a_2 \wedge b_2, \ldots, a_n \wedge b_n) \), where \( a_i \wedge b_i = \min(a_i, b_i) \) for each \( i \).

\( \mathcal{F} \subseteq \mathcal{M}(s_1, s_2, \ldots, s_n) \) is called a \( t \)-intersecting family if, for any \( a, b \in \mathcal{F}, a_i \wedge b_i > 0 \) for at least \( t \) 's. \( \mathcal{F} \) is called an \( \mathcal{P}_t \)-intersecting family if, for any \( a, b \in \mathcal{F}, a_i \wedge b_i \geq p \) for at least \( t \) 's.

What are the maximum \( \mathcal{P}_t \)-intersecting families in \( \mathcal{M}(s_1, s_2, \ldots, s_n) \)?

So far, results on the maximum \( t \)-intersecting families in \( \mathcal{M}(s_1, s_2, \ldots, s_n) \) have been obtained for a few \( t \)'s.

When \( t = 1 \), the maximum intersecting families in \( \mathcal{M}(s_1, s_2, \ldots, s_n) \) are known (see [1,5-7] and [10]). However, when \( t > 1 \), we have known little (also see [1] and [10]).

Definition 2. Let \( \mathcal{F} \subseteq \mathcal{M}(s_1, s_2, \ldots, s_n) \). \( \mathcal{F} \) is called a greedy \( \mathcal{P}_t \)-intersecting family if \( \mathcal{F} \) is an \( \mathcal{P}_t \)-intersecting family and \( W_p(A) \geq W_p(B + A^c) \) for any \( A \in S_p(\mathcal{F}) \) and any \( B \subseteq A \) with \( |B| = t - 1 \). \( \mathcal{F} \) is called a proper greedy \( \mathcal{P}_t \)-intersecting family if it is a greedy \( \mathcal{P}_t \)-intersecting family and \( W_p(A) > W_p(B + A^c) \) for any \( A \in S_p(\mathcal{F}) \) and any \( B \subseteq A \) with \( |B| = t - 1 \), where \( A^c = \{1, 2, \ldots, n\} - A \).

Comment. If \( W_p(A) \geq W_p(B + A^c) \) for any \( A \in S_p(\mathcal{F}) \) and any \( B \subseteq A \) with \( |B| = t - 1 \), then \( W_p(A) \geq W_p(B + A^c) \) for any \( A \in S_p(\mathcal{F}) \) and any \( B \subseteq A \) with \( |B| \leq t - 1 \).

If \( s_1 = s_2 = \cdots = s_n = s \) and \( p = 1 \), Engel and Frankl obtained the maximum \( t \)-intersecting families of multisubsets in [4]. Wu generalized Engel and Frankl's result in [9].

The above greedy property naturally comes from that of intersecting families. For any subset \( A \), the support of maximum intersecting families always contains the greatest weight one between \( A \) and \( A^c \) (see [1] and [10]). This is the greedy property of intersecting families.

For an \( \mathcal{P}_t \)-intersecting family \( \mathcal{F} \), we have that if \( A \in S_p(\mathcal{F}) \), then \( B + A^c \notin S_p(\mathcal{F}) \) for all \( B \subseteq A \) with \( |B| = t - 1 \). If we impose the property, that \( W_p(A) \geq W_p(B + A^c) \) for any \( A \in S_p(\mathcal{F}) \) and any \( (t - 1) \)-subset \( B \) of \( A \), on \( \mathcal{F} \), we then obtain a greedy \( \mathcal{P}_t \)-intersecting family in \( \mathcal{M}(s_1, s_2, \ldots, s_n) \).

Greedy \( \mathcal{P}_t \)-intersecting families form a special collection of \( t \)-intersecting families. In many cases, maximum greedy \( \mathcal{P}_t \)-intersecting families may reduce to maximum \( t \)-intersecting families. Our discussions for greedy \( \mathcal{P}_t \)-intersecting families may be a key to obtain sharp upper bounds of \( |\mathcal{F}| \) for \( t \)-intersecting families \( \mathcal{F} \) of finite sequences, which is still a hard problem.

From Definitions 1 and 2, we can see that when \( t = 1 \), maximum \( t \)-intersecting families of finite sequences are equivalent to maximum greedy \( t \)-intersecting families of finite sequences.

In this paper, we discuss the greedy \( \mathcal{P}_t \)-intersecting families in \( \mathcal{M}(s_1, s_2, \ldots, s_n) \). The discussions include three parts: \( \mathcal{P}_t \)-regular subsets, a \((t+1)\)-intersecting family of \( \mathcal{P}_t \)-greedy subsets, and the main theorems and their proofs. In our discussion, we always assume \( 2p \leq s_i \) for all \( t \) and \( s_1 > s_2 > \cdots > s_n \).

2. \( \mathcal{P}_t \)-Regular Subsets

In this section, we discuss \( \mathcal{P}_t \)-regular subsets of \( \mathcal{X} \), which are useful for us to obtain greedy \( \mathcal{P}_t \)-intersecting families of finite sequences. The collection of all \( \mathcal{P}_t \)-regular subsets is partitioned into a collection of pairs of \( \mathcal{P}_t \)-regular subsets. With this partition, we can construct a \((t+1)\)-intersecting family of \( \mathcal{P}_t \)-greedy subsets.