On the Convolution of Logistic Random Variables\(^1\)

By E.O. George, Nigeria\(^2\) and G.S. Mudholkar, Rochester\(^3\)

Abstract: An expression is obtained for the distribution of a convolution of independent and identically distributed logistic random variables by directly inverting the characteristic function. This distribution is shown to be closely approximated by a student's \(t\) distribution when both distribution are standardized. Moreover, by showing that some of the analytic simplicity and statistical properties that are manifest in the single logistic also obtain in the convolution, an application of the convolution as a dose-response curve in the bio-assay problem is suggested.

1. Introduction

The logistic distribution function

\[
F_{\mu, \sigma}(x) = \left[ 1 + \exp \left( -\frac{\sigma}{\pi \sigma} \frac{x - \mu}{\sqrt{3}} \right) \right]^{-1}, \quad -\infty < x < \infty. \tag{1.1}
\]

\(-\infty < \mu < \infty, \sigma > 0\), which has mean \(\mu\) and variance \(\sigma^2\), is remarkably similar to a normal distribution function with the same mean and variance. Moreover, when \(\mu = 0\) and \(\sigma^2 = \pi/3\), the distribution function

\[
F(x) = \left[ 1 + e^{-x} \right]^{-1}, \quad -\infty < x < \infty, \tag{1.2}
\]

satisfies the analytically simple relation

\[
f(x) = F(x) \left[ 1 - F(x) \right] \tag{1.3}
\]

with the density function \(f(x) = e^{-x}/(1 + e^{-x})^2\) and a linear log of odds ratio

\[
x = \log \left[ F(x)/(1 - F(x)) \right]. \tag{1.4}
\]

Largely because of these reasons, the logistic distribution is widely employed as a substitute for the normal distribution in applications such as bioassay and quantal response data problems. Berkson [1944] who introduces the term logit for the log of odds ratio, later, [1955], uses it to obtain minimum logit \(\chi^2\)-estimates for the loca-
tion and scale parameters of the logistic dose response curve. Other authors, notably Cox [1970] have also written extensively about this particular application of the logistic distribution.

The importance of the logistic in the modelling of stochastic phenomena has resulted in numerous other studies involving probabilistic and statistical aspects of the distribution. For example, Gumbel [1944], Gumbel/Keeney [1950] and Talacko [1956] show that it arises as a limiting distribution in various situations; Birnbaum/Dudman [1963], Gupta/Shah [1965] study its ordered statistics. Many other authors, e.g. Antle/Klimko/Harkness [1970], and Tartar/Clark [1965] investigate inference questions about its parameters. A good summary of results about the logistic distribution is found in Chapter 22 of Johnson/Kotz [1970].

As might be expected, because of the similarity between the logistic and the normal distributions, the sample mean and sample variance, the moment estimators of $\mu$ and $\sigma^2$, are effective tools for statistical decisions involving the logistic distribution. Antle/Klimko/Harkness [1970] give a function of the mean as a confidence interval estimate of $\mu$ when $\sigma^2$ is known. Schaffer/Sheffield [1973] show that in terms of the mean squared error the moments estimators of $\mu$ and $\sigma^2$ are as good as their maximum likelihood estimators. The fact that the distribution of the sample mean has monotone likelihood ratio with respect to $\mu$ when $\sigma$ is known is used by Goel [1975] to obtain a uniformly most accurate confidence interval for $\mu$ and a uniformly most powerful test for one-sided hypotheses involving $\mu$. The sampling distribution of the mean is a primary requirement for these statistical purposes. The papers due to Antle/Klimko/Harkness [1970] and Tartar/Clark [1965] both contain Monte Carlo'ed results for this distribution. Goel [1975] obtains an expression for the distribution function of the sum of independent and identically distributed (i.i.d.) logistic variates by using the Laplace transform inversion method for convolutions of Polya type functions, a technique developed by Schoenberg [1953] and Hirschman/Widder [1955]. He uses this approach because a direct inversion of the characteristic function leads him to a very slowly converging series for its distribution function. In section 2 of this paper we demonstrate that the characteristic function can be directly inverted and obtain a neat expression for the distribution function. In section 3 we show that a standardized student's $t$ distribution provides an excellent approximation for the distribution of the logistic sample mean. In section 4 it is shown that a function of the convolution satisfies equations (1.3) and (1.4). In view of this result, possible use of the convolution of the logistic distribution as a dose-response curve is discussed.

2. Distribution Function of the Sum of i.i.d. Logistic Variates

Let $X_1, X_2, \ldots, X_n$ be i.i.d. with distribution function

$$F(x) = [1 + \exp (-x)]^{-1},$$

(2.1)

and let

$$Z = \frac{n}{\sum_{j=1}^{n} X_j}.$$  

(2.2)