A LIGHT SUBSYSTEM INSIDE A GRAVITATING SPHEROID

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We establish the possible shapes of the equilibrium configurations of a light subsystem with internal flux of matter of constant vorticity inside a homogeneous gravitating spheroid. We determine the geometric and kinematic properties of ellipsoidal and hyperboloidal figures, and also their stability vis-a-vis second-form perturbations.

1. Introduction. In the theory of equilibrium shapes of rotating gravitating masses the steady state is reached through balancing of the forces of rotational inertia, pressure, and self-gravitation of the system [1, 2]. If the rotating mass is imbedded in a larger gravitating system, it is necessary to take account of the external gravitating system as well when studying questions of equilibrium and stability of equilibrium shapes for the imbedded mass. In the context of such a formulation of the problem a theory of equilibrium shapes of imbedded gravitating subsystems has been developed [3-5] in which the classical ellipsoidal equilibrium shapes have been generalized and new shapes have been obtained that have no analogs for single figures. Besides its purely theoretical interest, the results of this work are valuable from the point of view of astronomical applications, to explain a number of dynamic and kinematic properties of various subsystems of spiral and elliptic galaxies [3, 5, 6, 7].

The presence of an external gravitating mass (principal body) makes it possible to pose the problem of the dynamics of imbedded “light” subsystems whose self-gravitation is negligibly small compared to the gravitation of the principal body. We shall obtain a condition for the applicability of the “light” subsystem approximation by comparing the forces acting on a test particle from the imbedded subsystem and the principal body. Obviously the ratio of these forces is of the same order as the ratio of the characteristic masses of these subsystems:

\[ \frac{M}{M^*} \approx \frac{\rho h}{\rho^* h^*} \gg 1, \] (1.1)

where \( \rho, \rho^*, h, h^* \) and \( h \) are the characteristic three-dimensional mass densities and the thicknesses (along the axis of rotation) of the imbedded subsystem and the principal body respectively. Polyachenko and Fridman [8] have proved rigorously that \( h/h^* \approx c_{11}/c_{*11} \), where \( c_{11} \) and \( c_{*11} \) are the dispersions of the velocities of the stars of the respective subsystems along the axis of rotation of the system. If we regard the gas component of a galaxy as a light subsystem, then \( c_{11} \) is the speed of sound. Consequently, the condition for lightness of the imbedded gas subsystem assumes the form

\[ \frac{\rho^*}{\rho} \ll \frac{h}{h^*} \approx \frac{c_{11}}{c_{*11}}, \] (1.2)

from which it follows that for approximately equal three-dimensional mass densities a strongly oblate subsystem inside a weakly oblate system is light and the effects of its self-gravitation can be neglected in studying its dynamics.

The conditions for lightness are well satisfied for a layer of the interstellar medium in galaxies, for individual strongly oblate subsystems of stars, and so forth.

The study of the dynamics of a light subsystem is of interest from the point of view of another important problem. As of the present the problem of establishing the true geometry of elliptic galaxies has not been definitively settled. In fact for a number of elliptic galaxies, such as, for example, NGC 5128, 5266, 3107, 5363, 1947 and so forth, a layer of diffuse matter has been detected. The study of the dynamics and kinematics of a light subsystem and the establishment of the possible shapes of the equilibrium figures and conditions for their stability inside stellar systems of different ellipsoidal geometries may yield valuable information on the shapes of the stellar subsystems of these galaxies [6, 7, 9].

2. Equilibrium of a light subsystem with internal flows of matter. Inside a gravitating principal body in the shape of a homogeneous spheroid we consider the equilibrium of a light subsystem having internal steady-state flows of matter with constant vorticity in a coordinate system rotating with angular velocity $\Omega$:

$$u_\alpha = -Q_{\alpha \beta} x_\beta; \quad \alpha, \beta = 1, 2, 3,$$

(2.1)

where $x_\beta$ are the coordinates of a Cartesian coordinate system with origin at the center of the system, and

$$Q_{\alpha \beta} = \lambda \Omega \frac{a_\alpha}{a_\beta} \epsilon_{\alpha \beta \gamma}. \quad (2.2)$$

In the plane of rotation ($X_1, X_2$) the streamlines of (2.1) are similar and concentric ellipses with semiaxes proportional to $a_1$ and $a_2$ along which particles move with velocity $\nu = \lambda \Omega$. Positive values of $\lambda$ correspond to motions of particles along these streamlines in the direction of rotation of the subsystem, negative values correspond to motions opposite to the rotation of the subsystem, and $\epsilon_{\alpha \beta \gamma}$ is the Levi-Civita tensor.

In the presence of the internal flows (2.1) the equilibrium of a light subsystem will be determined by the Coriolis force along with the axisymmetric forces of the spheroidal principal body, the centrifugal forces, and the pressure. For that reason it turns out that there may exist nonaxisymmetric equilibrium figures of a light subsystem.

The relative equilibrium of a light subsystem is determined by the equation

$$\frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{1}{\rho} \frac{\partial V_\star}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_i} \left[ (\Omega \mathbf{x})^2 - 2\epsilon_{ijkl} u_k \Omega_l \right], \quad (2.3)$$

where summation is performed on repeated subscripts, $V_\star$ is the gravitational potential of the external spheroid at an interior point [1]

$$V_\star(\mathbf{x}) = -A_\star(x_1^2 + x_2^2) - A_3 x_3^2, \quad (2.4)$$

the unit of time is $(\pi \rho_\star)^{-1/2}$, and $\rho$ and $P$ are the partial mass density and the partial mass pressure of the light subsystem.

Integrating Eqs. (2.3) taking account of the relations (2.1), (2.2), and (2.4) we obtain the following expression for the partial pressure of the light subsystem

$$P(\mathbf{x}) = P_c \left\{ 1 - \frac{x_3^2}{a_3^2} - \frac{2A_\star - \Omega^2(1 + \lambda^2 + 2\lambda a_2/a_1)}{2a_3^2 A_3^*} x_1^2 - \frac{2A_\star - \Omega^2(1 + \lambda^2 + 2\lambda a_1/a_2)}{2a_3^2 A_3^*} x_2^2 \right\}, \quad (2.5)$$

where the constant of integration is chosen as $P_c = \pi \rho_\star \rho a_3^2 A_3^*$.

Let us now consider the possible shapes of the equilibrium figures of a light subsystem.

2a. Triaxial ellipsoids. It can be seen from the expression (2.5) just obtained for the partial pressure that when the conditions

$$\Omega^2 (1 + \lambda^2 + 2\lambda a_2/a_1) = 2A_\star - 2\frac{a_3^2}{a_1^2} A_3^*, \quad (2.6)$$

$$\Omega^2 (1 + \lambda^2 + 2\lambda a_1/a_2) = 2A_\star - 2\frac{a_3^2}{a_2^2} A_3^*, \quad (2.7)$$

hold, equilibrium of a light subsystem is possible in the form of a triaxial ellipsoid with semiaxes $a_1$, $a_2$, and $a_3$.

Relations (2.6) and (2.7) give physical and geometric properties of the possible triaxial-ellipsoidal equilibrium figures of a light subsystem inside a gravitating homogeneous spheroid.

Eliminating the angular velocity from (2.6) and (2.7), we obtain an equation for $\lambda$:

$$\lambda + 2\lambda \frac{a_1 a_2}{a_3^2} \frac{A_3^*}{A_3^*} + 1 = 0, \quad (2.8)$$