SMOOTHING COMPLEMENTS AND RANDOMIZED SCORE FUNCTIONS

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Abstract

This paper establishes connections between two derivative estimation techniques: infinitesimal perturbation analysis (IPA) and the likelihood ratio or score function method. We introduce a systematic way of expanding the domain of the former to include that of the latter, and show that many likelihood ratio derivative estimators are IPA estimators obtained in a consistent manner through a special construction. Our extension of IPA is based on multiplicative smoothing. A function with discontinuities is multiplied by a smoothing complement, a continuous function that takes the value zero at a jump of the first function. The product of these functions is continuous and provides an indirect derivative estimator after an appropriate normalization. We show that, in substantial generality, the derivative of a smoothing complement is a randomized score function: its conditional expectation is a derivative of a likelihood ratio. If no conditional expectation is applied, derivative estimates based on multiplicative smoothing have higher variance than corresponding estimates based on likelihood ratios.

1. Introduction

There are two classes of methods for estimating derivatives of expectations without finite differences: those based on infinitesimal perturbation analysis (IPA) and those based on the likelihood ratio or score function method. (see Ho [11], Glynn [8], Reiman and Weiss [14], Rubinstein [16].) This paper establishes fundamental connections between these two classes by systematically expanding the domain of the former to include that of the latter. We show that a broad range of likelihood ratio derivative estimators are IPA estimators obtained in a consistent manner through a special construction.

Let us begin with some background on the methods. Let \((X, \mathcal{A})\) be a measurable space supporting a family \((\mathcal{P}_\theta, \theta \in \Theta)\) of probability measures. Let \(\Gamma\) be a real-valued, measurable function on \(X\) and let

\[
g(\theta) = \int_X \Gamma(x) P_\theta(dx).
\]

We seek estimators \(g'(\theta) = dg(\theta)/d\theta\) when this derivative exists.
In IPA, one first defines (often only implicitly) a family \( \{X(\theta), \theta \in \Theta\} \) of random elements of \( X \), on a common probability space \( (\Omega, \mathcal{F}, P) \) which does not depend on \( \theta \). Each \( X(\theta), \theta \in \Theta \), has law \( \mathcal{P}_\theta \). One then lets \( \gamma(\theta) = \Gamma(X(\theta)); \) naturally, \( \mathbb{E}[\gamma(\theta)] = g(\theta) \), where \( \mathbb{E} \) is expectation on \( (\Omega, \mathcal{F}, P) \). An IPA estimator of \( g'(\theta) \) is then given by

\[
\gamma'(\theta) = \mathbb{E}[\gamma(\theta)'] = g'(\theta).
\]

The likelihood ratio method is based on a different point of view. Suppose that \( \{\mathcal{P}_\theta, \theta \in \Theta\} \) are mutually absolutely continuous; i.e.

\[
\mathcal{P}_\theta(A) > 0 \iff \mathcal{P}_\theta(A) > 0, \quad \forall A \in \mathcal{A}, \forall \theta_1, \theta_2 \in \Theta. \tag{1}
\]

This ensures the existence of a likelihood ratio \( L(x, \theta_2, \theta_1) = (d\mathcal{P}_{\theta_2}/d\mathcal{P}_{\theta_1})(x) \), for all \( \theta_1, \theta_2 \), and allows us to write

\[
g(\theta + h) - g(\theta) = \int \Gamma(x) \mathcal{P}_{\theta + h}(dx) - \int \Gamma(x) \mathcal{P}_\theta(dx).
\]

\[
= \int \Gamma(x)[L(x, \theta + h, \theta) - 1] \mathcal{P}_\theta(dx). \tag{2}
\]

Suppose, now, that \( L(x, \cdot, \theta_1) \) is differentiable for all \( x \) and \( \theta_1 \), and denote partial derivative (with respect to the second argument) by \( \partial_2 L(x, \theta, \theta) \). When evaluated at \( \theta_1 = \theta, \partial_2 L(x, \theta, \theta) \) is called the score function for \( \{\mathcal{P}_\theta, \theta \in \Theta\} \). Dividing both sides of (2) by \( h \) and letting \( h \to 0 \), we get

\[
g'(\theta) = \int \Gamma(x) \partial_2 L(x, \theta, \theta) \mathcal{P}_\theta(dx),
\]

provided the interchange of limit and integral is valid. Under this condition, it follows that \( \Gamma(X(\theta)) \partial_2 L(X, \theta, \theta) \) is an unbiased estimator of \( g'(\theta) \). When \( \{\mathcal{P}_\theta, \theta \in \Theta\} \) is given by a collection \( \{p_n(\theta), n = 0, 1, 2, \ldots, \theta \in \Theta\} \) of probability mass functions, the score function takes the form \( p'_X(\theta)/p_X(\theta) \). When the measures are given by densities \( \{f(x, \theta), \theta \in \Theta, x \in X\} \), the score function is \( \partial_2 f(X, \theta) | f(X, \theta) \), where \( \partial_2 f = \partial f/\partial \theta \).

IPA and the likelihood ratio method both require an interchange of limit and integral, and therefore are only valid when appropriate regularity conditions are in effect. The specific conditions needed for the two methods are different, so – in their simplest forms – the methods work on different classes of problems (though there is overlap). It is reasonable to combine the methods to obtain greater applicability. The derivations above were based on placing all the dependence on \( \theta \) in the random elements \( X(\theta) \), or all of it in measures \( \mathcal{P}_\theta \). Through judicious choice of an intermediate space \( (\bar{X}, \mathcal{A}) \) it is generally possible to express \( g(\theta) \) as