SCATTERING ON UNIVALENT GRAPHS FROM
THE L-FUNCTION POINT OF VIEW

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The scattering process on multiloop infinite \((p+1)\)-valent graphs is studied. These graphs are discrete spaces of a constant negative curvature, being quotients of a \(p\)-adic hyperbolic plane over free-acting discrete subgroups of the projective group \(PGL(2, \mathbb{Q}_p)\). They are, in fact, identical to \(p\)-adic multiloop surfaces. A finite subgraph containing all loops is called the reduced graph \(T_{\text{red}}\); the L-function is associated with this finite subgraph. For an infinite graph, we introduce the notion of spherical functions. They are eigenfunctions of a discrete Laplace operator acting on the graph. In scattering processes, we define the s-matrix and the scattering amplitudes \(c_i\), imposing the restriction \(c_i = A_{\text{ret}}(u)/A_{\text{adv}}(u) = \text{const}\) for all vertices \(u \in T_{\text{supp}}\). \(A_{\text{ret}}\) and \(A_{\text{adv}}\) are retarded and advanced branches of a solution to the eigenfunction problem and \(T_{\text{supp}}\) is a support domain for scattering centers. Taking the product over all \(c_i\), we obtain the determinant of the scattering matrix, which is expressed as a ratio of two L-functions: \(C = \frac{L(c_+)}{L(c_-)}\). Here the L-function is the Ihara-Selberg function depending only on the forum of \(T_{\text{red}}, a_{\pm} = t/2p \pm \sqrt{t^2/4p^2 - 1}/p\), \(t - 1\) being the eigenvalue of the Laplacian. We present a proof of the Hashimoto-Bass theorem, the expressing L-function \(L(u)\) of any finite graph via the determinant of a local operator \(\Delta(u)\) acting on this graph. Numerous examples of L-function calculations are presented.

0. Introduction

This paper is based on the author’s paper [1]. In that paper, we investigated scattering processes on infinite graphs that represent spaces of a constant negative curvature. This means that for a given graph there exists a unique natural number \(p \geq 1\) such that the valencies of all vertices of the graph are equal to \(p + 1\). These graphs for prime \(p\) have the meaning of homogeneous spaces for the \(p\)-adic projective group \(PGL(2, \mathbb{Q}_p)\) factorized, first, by its maximal compact subgroup \(PGL(2, \mathbb{Z}_p)\), and, second, by some discrete Schottky group \(\Gamma\). Our aim is to find a proper analog of the Selberg trace formula [2] for such surfaces. These graphs are also often referred to as multiloop \(p\)-adic surfaces since it has been shown in [3, 4] that for prime \(p\) a discrete Laplacian acting on these graphs yields the proper scattering amplitudes of the \(p\)-adic string. In what follows, we look for an intermediate chain between the following domains of the graph theory:

1. Multiloop \(p\)-adic surfaces were introduced in [3]. They can be explicitly presented as a set of (infinite) graphs \(T_g\) with a finite number \(g\) of primitive cycles, \(g\) being the genus of the surface. (We often omit the index \(g\).) For a fixed prime number \(p\), the valences of all vertices of the graph are the same — \(p + 1\). For \(g = 0\), we get a tree graph which we denote as \(X, X \simeq PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)\), and the vertices of \(X\) correspond to prime ideals. It is possible to introduce a metric on this space, defining the distance between two points. Explicitly, this distance coincides with the distance in the tree \(X\) — the number of edges in the shortest way connecting these two points.

Let us now factorize the tree \(X\) by a discrete freely acting finitely generated subgroup \(\Gamma_g\) of \(PGL(2, \mathbb{Q}_p)\), \(g\) being the number of generating elements. The graph obtained \(T_g = X/\Gamma_g\) is again a univalent graph, but it contains \(g\) loops. The graph \(X\) is a universal covering for all graphs \(T_g\) with the same generating number \(p\). We shall consider linear spaces of functions \(C_0\) and \(C_1\) depending, respectively, on the vertices \(x_i\) and the oriented edges \(e_j\) of the graph \(T\). A Laplace operator \(\Delta\) acts on the space \(C_0\), \(\Delta f_0 = \sum_{\text{neigh}} f_i - (p + 1)f_0\), where the sum runs over all neighbors of point \(x_0, f_i = f(x_i)\). It is useful to release from the graph \(T\) its finite "closed" part — the reduced graph \(T_{\text{red}}\) containing all internal loops. The valences of \(T_{\text{red}}\) vertices can be arbitrary \((\leq p + 1)\). In [4], the string theory for such graphs was developed, and it was demonstrated...
that when calculating the corresponding amplitudes, all crucial ingredients of the string theory, such as prime forms, Schottky groups, etc., have their proper p-adic analogies. It is worth noting that, due to Schottky uniformization, all these p-adic surfaces are surfaces of constant negative curvature.

2. For ordinary closed Riemann surfaces of constant (negative) curvature, the Selberg trace formula is well known [2], which establishes an explicit relation between determinants of the Laplace operators and zeta-functions (or Ihara–Selberg L-functions) of such surfaces. An L-function is defined as follows:

\[ L(u) = \prod_{\{\gamma\}} (1 - u^{l(\gamma)})^{-1}, \tag{0.1} \]

where the product runs over all primitive (i.e., without subperiods) closed geodesics on the surface, \( l(\gamma) \) being the lengths of the geodesics in the constant negative curvature metric. However, for the noncompact case where the spectrum of the Laplacian contains a continuous part, no good analog of the Selberg formula is known. What we are going to do is to obtain it at least for the case of noncompact discrete spaces.

3. In [5], an analog of the Selberg trace formula has been obtained for the case of a finite graph, again with the same, say, \( p+1 \), valence for all vertices. Recently, in a series of elegant papers by Hashimoto [6] and Bass [7], this approach was enlarged to the case of an arbitrary finite graph:

\[ L(u) = (1 - u^2)^{|T| - |E|} \det^{-1}(1 + u^2Q - uM_1) \equiv (1 - u^2)^{1-\theta} \det^{-1} \Delta(u), \tag{0.2} \]

where \( L(u) \) is the corresponding L-function (0.1) — an infinite product over all primitive closed paths in the finite graph, \( M_1 \) is the operator of summation over all neighbors, and \( Q \) is a new operator that counts valences: \( Qx = qx \) if the vertex \( x \) has \( q + 1 \) neighbors; here \(|T|\) and \(|E|\) are total number of vertices and edges of the graph, respectively. Note, however, that this operator, \( \Delta(u) \), is neither Laplacian nor does it commute with \( \Delta \) (they coincide only for \( u = 1 \)). Therefore, a straightforward application of Selberg’s ideology to this case seems to be impossible. (For a detailed description of these results and their different applications, see [8].)

4. The p-adic multiloop graphs are noncompact, in contrast both to the closed Riemann surfaces and to finite graphs. A spherical function technique was used in this case. Originally, spherical functions \( F \) are such eigenfunctions of the Laplace–Beltrami operator that depend only on the distance from a selected point \( x_0 \) (the center). Since a Laplacian is a second-order operator, we always have two branches of the solution (at a distant point) expanded as \( a_+^+ \) and \( a_-^- \), \( d \) being the distance to the center. Resolving the eigenvalue problem at the central point, we fix the ratio of coefficients \( a_+ / a_- \) standing by these two branches. If we choose \( F(x_0) = 1 \), then \( a_+ \) and \( a_- \) become Harish-Chandra coefficients, and \( c = a_+ / a_- \) is the scattering amplitude of the s-wave. Spherical functions have been found for the scattering on a quantum hyperplane [9] as well as for the scattering on a p-adic hyperbolic plane [10]. The \( S \)-matrices obtained are closely related to the partition functions of the \( XXZ \) model, and a lot of nice but somewhat mysterious relations between them have been obtained in the series of papers by Freund and Zabrodin [11].

In [1], an analog of the spherical function for the multiloop case was introduced. The main idea is as follows. Note that any linear superposition of spherical functions with the same eigenvalue \( t \) but different scattering centers is again an eigenfunction. In order to get an eigenfunction of the Laplace operator on the factorized tree \( T = X/\Gamma_g \), we choose a source distribution function \( s(x) \) on the tree \( X \) such that for every \( \gamma \in \Gamma_g \) and every \( x \in X \), \( s(\gamma x) = s(x) \). Then the whole eigenfunction is periodic under the action of \( \Gamma_g \). Moreover, we choose a finite domain \( T_{\text{supp}} \subset T \) and consider only \( s(x) \) such that \( \text{supp} s(x) \subset T_{\text{supp}} \). We denote this space of eigenfunctions by \( \mathcal{F}(t, T_{\text{supp}}) \). Inside \( T \) there is a unique minimal finite connected subgraph containing all loops, which is called the “reduced graph,” \( T_{\text{red}} \). This graph contains all the information about the “geometrical structure” of \( T \). We always assume \( T_{\text{red}} \subset T_{\text{supp}} \).

Each function \( F(x) \in \mathcal{F}(t, T_{\text{supp}}) \) may be presented as a sum of retarded and advanced wave functions:

\[ F(x) = A_{\text{ret}}(u)\alpha_+^d(x, u(x)) - A_{\text{adv}}(u)\alpha_-^d(x, u(x)) \tag{0.3} \]

Here, as before, \( \alpha_+ \) and \( \alpha_- \) are two fixed complex numbers depending only on the eigenvalue \( t \) of the Laplacian \( D \), which acts on the whole graph \( T \): \( Df(x) = (t - p - 1)f(x) \), and on the initial prime number \( p \),