DARBOUX TRANSFORMATION, FACTORIZATION, AND SUPERSYMMETRY IN ONE-DIMENSIONAL QUANTUM MECHANICS

V.G. Bagrov and B.F. Samsonov

We introduce an \( N \)-order Darboux transformation operator as a particular case of general transformation operators. It is shown that this operator can always be represented as a product of \( N \) first-order Darboux transformation operators. The relationship between this transformation and the factorization method is investigated. Supercharge operators are introduced. They are differential operators of order \( N \). It is shown that these operators and super-Hamiltonian form a superalgebra of order \( N \). For \( N = 2 \), we have a quadratic superalgebra analogous to the Sklyanin quadratic algebras. The relationship between the transformation introduced and the inverse scattering problem in quantum mechanics is established. An elementary \( N \)-parametric potential that has exactly \( N \) predetermined discrete spectrum levels is constructed. The paper concludes with some examples of new exactly soluble potentials.

1. Introduction

At present several methods of constructing exactly soluble potentials of the one-dimensional Schrödinger equation are known. First is the Darboux method [1] based on applying a first-order differential operator to the solution of some known problem. Second is the factorization method [2, 3] applied recently for constructing potentials with oscillator spectrum [4], and also the high-dimensional oscillator and Coulomb potential [5]. Third is the method of integral transformations, which forms the basis of the inverse scattering method. These transformations are derived from the solution of Gelfand-Levitan [7] or Marchenko [8] equations and are known as the Abraham–Moses–Pursey transformations [9–11]. Recently they have been generalized to the nonstationary Schrödinger equation [12–13]. The first two methods are closely related [14] to the problems of supersymmetry of the Schrödinger equation (see, for example, [5, 15]).

In this paper, starting from the general transformation operators introduced by Delsart [7, 16], we define a Darboux transformation operator of order \( N \). It is shown that this operator can always be represented as the product of \( N \) first-order Darboux transformations (i.e., ordinary Darboux transformations [1]). Recently this statement has been proved for operators with equidistant spectrum [17]. The relationship between this transformation and the factorization method is studied. Supercharge operators are introduced which are of order \( N \) on the derivative. It is found that these operators together with the super-Hamiltonian form a superalgebra of order \( N \). When \( N = 2 \), we have the quadratic superalgebra; it is analogous to the Sklyanin quadratic algebras [18] which are used for the description of the dynamical symmetry of the Schrödinger equation [19]. Finally, some new examples of exactly soluble potentials are given.

2. \( N \)-order Darboux transformation

Assume that we know the general solution \( \psi_E^0(x) \) of the second-order differential equation

\[
H_0\psi_E(x) = E\psi_E(x), \quad H_0 = -d^2/dx^2 + V_0(x), \quad x \in [a, b],
\]

(2.1)

at any value of parameter \( E \) (here \( V_0(x) \) is some reasonably smooth function; the interval \([a, b]\) can be infinite). The set of solutions of Eq. (2.1) endowed with topological structure form a topological space which will be denoted by \( T_0 \). The topological space of all reasonably smooth functions on \([a, b]\) is designated as \( T \). Following Delsart [16] let us introduce
Definition 2.1. A linear differential operator $L_N$ of order $N$ acting from $T_0$ to $T_0$ is called an $N$-order Darboux transformation operator if it is reversible and the following conditions are fulfilled:

1) $L_N$ and $L_N^{-1}$ are continuous;
2) the following equalities hold:

\[
L_N H_0 = H_N L_N, \quad H_N = -d^2/dx^2 + V_n(x), \quad V_n(x) \in T. \tag{2.2} \]

According to relation (2.2), the general solution $\varphi^0_E(x)$ of the equation

\[
H_N \varphi_E(x) = E \varphi_E(x), \quad x \in [a, b], \tag{2.4} \]

is related to the solution $\psi_E^0(x)$ of (2.1):

\[
\varphi^0_E(x) = L_N \psi^0_E(x). \tag{2.5} \]

If we denote

\[
A_N(x) = H_N - H_0 = V_N(x) - V_0(x), \tag{2.6} \]

then equality (2.2) takes the form

\[
[L_N, H_0] = L_N H_0 - H_0 L_N = A_N(x) L_N. \tag{2.7} \]

Given operator $H_0$, equality (2.7) can be treated as an equation in the function $A_N(x)$ (this function determines the potential $V_N(x)$ of a new Schrödinger equation (2.4)), and in the operator $L_N$ connecting the solutions $\varphi_E(x)$ of Eq. (2.4) with the solutions $\psi_E(x)$ of Eq. (2.1) at the same value of spectral parameter $E$.

When $N = 1$, Eq. (2.7) leads to the ordinary Darboux transformation [1]:

\[
L_1 \equiv L = L_0(C, x) - d/dx, \quad L_0(C, x) = \psi'_C(x)/\psi_C(x), \quad A_1(x) \equiv A(C, x) = -2L_0'(C, x), \tag{2.8} \]

where the prime denotes the derivative on $x$, and $\psi_C(x)$ is some solution of Eq. (2.1) at $E = C$ ($C$ is an arbitrary constant).

Operator $L$ is completely determined by the function $\psi_C(x)$. In connection to this topic, let us introduce the following

Definition 2.2. The functions $\psi_1(x), \ldots, \psi_k(x)$ are called transformation functions if they uniquely determine the Darboux transformation operator $L_N$.

The function $\psi_C(x)$ is the Darboux transformation function of the first order.

Remark 2.1. For the function $\psi_C(x)$ in formulas (2.8) one can take the general solution $\psi^0_C(x)$ of Eq. (2.1) which depends on two arbitrary constants $C$ and $\alpha$ (here $\alpha$ is a constant of the linear combination of two linearly independent solutions of Eq. (2.1) when $E = C$ is fixed). The operator $L_1$ also depends on these two parameters.

Theorem 2.1. The operator $L_N$ can always be represented as the product $L_N = L_1^{(1)} \cdot L_1^{(2)} \cdots L_1^{(N)}$, where $L_1^{(k)}$ ($k = 1, \ldots, N$) is the first-order Darboux transformation operator. It is defined by relations (2.8) from the solutions of the Schrödinger equation derived from the initial equation by the Darboux transformation of order $k - 1$.

Remark 2.2. In the particular case $A_N(x) = \text{const}$ this theorem have been proved in [17].

Let us sketch out the proof of Theorem 2.1. Operator $L_N$ depends on $N + 1$ functions $L_i^{(N)}(x)$: $L_N = \sum_{i=0}^{N} L_i^{(N)}(x) d^i/dx^i$. If all differentiation operators $d^i/dx^i$ in relation (2.7) are assumed to be linearly