ON THE SEMIBOUNDEDNESS OF δ-PERTURBATIONS OF THE LAPLACIAN SUPPORTED BY CURVES WITH ANGLE POINTS

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Elementary self-adjoint perturbations of the Laplacian supported by curves with singular angle points in $\mathbb{R}^3$ and $\mathbb{R}^4$ are studied. The perturbations are shown to be semibounded in $\mathbb{R}^3$ and not semibounded in $\mathbb{R}^4$. In the latter case semiboundedness may take place in subspaces with a given symmetry, as simple examples illustrate.

Introduction

Interactions focused on a set $\Sigma \subset \mathbb{R}^n$ of zero measure find many applications in mathematical physics. A general approach to the study of such interactions is the theory of self-adjoint operator extensions. Two problems within this theory are important from the point of view of applications: the existence and semiboundedness of the elementary extensions of the Laplacian, which are determined by $\delta$ interactions supported by $\Sigma$. The existence of such extensions depends, in a critical way, on the Hausdorff dimensions, and semiboundedness — on the smoothness of $\Sigma$. We are interested in the case where $\Sigma$ is the piecewise smooth curve with angle points and intersection points in $\mathbb{R}^n$, $n = 3, 4$. The case of smooth curves in three-dimensional space was considered in [1, 2]. In [3, 4] the case of brownian type nonregular curves for three and four dimensions was considered. Interest in dimensions three and four for piecewise curves arises from the fact that $n = 4$ is the maximum dimension for nontrivial $\delta$-perturbations to exist, and $n = 3$ is the minimum dimension where specifying them is connected with renormalization. Note that for brownian curves, the maximum dimension is five [4].

This paper considers the question of the semiboundedness of elementary extensions for typical representatives of curves with singular angle points: $\Sigma$ is a simple angle or a set of intersecting straight lines in $\mathbb{R}^3$ and $\mathbb{R}^4$. It is shown that semibounded elementary $\delta$-perturbations on them exist in $\mathbb{R}^3$ and are absent in $\mathbb{R}^4$. Hence, it follows that the dimension $n = 3$ is the largest where such perturbations can exist on a set of curves with angular points. As is shown in [4] for dimension $n = 3$, the $\delta$-perturbation on a brownian curve is also semibounded (see also [3]). The absence of semibounded $\delta$-perturbations in $\mathbb{R}^4$, which is discouraging, urges one to consider lower classes of extensions in the subspaces of functions with specified symmetries induced by the symmetry of $\Sigma$. Further it will be shown that in some of these subspaces, elementary extensions are semibounded, and in this case the condition of semiboundedness imposes conditions on the geometry of the curve. For example, for two straight lines intersecting at the zero point in $\mathbb{R}^4$, in a subspace of odd functions, the semiboundedness of elementary extensions depends in a critical way on the angle between the straight lines and takes place only under the condition that $\vartheta + \sin \vartheta \geq \pi/2$. This problem is closely related to the problem of three pointwise interacting quantum particles [5, 6], and to the problem of the spectrum for the Hamiltonian in $L^2(\mathbb{R}^3)$

$$H = (-\Delta + \mu)^{1/2} - \mu^{1/2} - \alpha r^{-1}, \quad \mu \geq 0.$$ 

As is known [7], it is self-adjoint and semibounded only for $\alpha \leq 2/\pi$. Another direction for solving the problem of stability is connected with the use of generalized $\delta$-perturbations, and a separate paper will be devoted to their consideration.
Let us concretely define the problem. Suppose that \( \Sigma \) is a piecewise smooth curve in \( \mathbb{R}^n, n = 3, 4 \), in a parametric representation \( x(t): I \subset \mathbb{R}^1 \rightarrow \mathbb{R}^n \), finite or infinite, all singularities of which consist only of a finite number of angle points and points of self-intersection at a nonzero angle. Denote a set of such points by \( \Sigma_0 \), and a set of their prototypes in \( I \) by \( I_0 \). The spaces of the functions on \( \Sigma \setminus \Sigma_0 \) and on \( I \setminus I_0 \) will be identified. In addition, we shall consider the sums of several curves as a curve with a complicated parameter. Parametrization is assumed to be such that |\( \dot{x}(t) \)| = 1, except for singular points.

The unperturbed Hamiltonian \( H_0 = -\Delta \) is a self-adjoint operator in \( L^2(\mathbb{R}^n) \) with its domain of definition being the Sobolev space \( H^2(\mathbb{R}^n) \). \( H^s(\mathbb{R}^n) \) denotes the space of generalized functions \( f \), whose Fourier prototypes \( \hat{f} \) are in \( L^2(\mathbb{R}^n) \). For \( \mu > 0 \), the Green function \( G^{(n)}_\mu(x, y) = (-\Delta + \mu)^{-1}(x, y) \) is the kernel of the resolvent of the operator \( H_0 \). We denote its trace on \( \Sigma \times \Sigma \) by \( G^{(n)}(t, s) \). We shall omit the space dimension index, if this does not lead to ambiguity.

On a set of smooth functions of compact support on \( C^\infty_0(\mathbb{R}^n) \) is defined the trace operation \( i_\Sigma f = f|_{\Sigma} \), continued for \( s - (n - 1)/2 > 0 \), as is known, till there is a continuous mapping of the space \( H^s(\mathbb{R}^n) \) in \( L^2(\Sigma) \approx L^2(I) \). The Hamiltonians of interactions supported by \( \Sigma \) are determined as the self-adjoint extensions of the symmetric operator \( \widetilde{H}_0 \), determined by the restriction of \( H_0 \) to

\[
D_{\widetilde{H}_0} = \{ f \in H^2(\mathbb{R}^n) \mid i_\Sigma f = 0 \}.
\]

For \( n \leq 4 \) we have a guarantee of the existence of such extensions different from \( H_0 \). They are separated out by restricting the self-adjoint operator \( \widetilde{H}_0^+ \):

\[
D_{\widetilde{H}_0^+} = \{ f = f_0 + G_\mu \ast F; \mu > 0, f_0 \in H^2(\mathbb{R}^n), F \in \mathfrak{S} \},
\]

\[
\widetilde{H}_0^+ f = H_0 f_0 - \mu G_\mu \ast F,
\]

by specifying a self-adjoint linear relation in the space \( \mathfrak{S} \). Here, \( \mathfrak{S} \) is the Hilbert space which arises following the completion of the space \( C^\infty_0(I \setminus I_0) \) relative to the norm \( \| W^{1/2} \cdot \|_{\mathfrak{S}_0} \), \( \mathfrak{S}_0 = L^2(I) \), \( W \) is the positive definite integral operator with kernel \( W(t, t') = G_\mu \ast G_\mu(x(t), x(t')) \). The sesquilinear form \( \langle \cdot, \cdot \rangle = \langle \cdot, W \cdot \rangle_{\mathfrak{S}_0} \) is a scalar product in \( \mathfrak{S} \). The kernel \( W(t, t') \) has no singularities in \( n = 3 \) and has a logarithmic singularity on a diagonal in \( n = 4 \). Everywhere we suppose that the condition

\[
\sup_{t \in I} \int W(t, t') \, dt' < \infty, \quad (D)
\]

providing for the boundedness of the operator \( W \), holds. It is the condition of the global behavior of the curve, and it holds for all cases considered by us. Under condition (D), by extending \( W \) continuously to the entire space \( \mathfrak{S} \), we equip the space \( \mathfrak{S}_0 \):

\[
\mathfrak{S}_+ \subset \mathfrak{S}_0 \subset \mathfrak{S} \quad \text{with} \quad \mathfrak{S}_+ = W \mathfrak{S}, \quad \mathfrak{S} = W^{-1} \mathfrak{S}_+.
\]

Here the operators \( W \) and \( W^{-1} \) are interpreted as isometries of the corresponding spaces. In the case of smooth curves in \( n = 3 \) we have \( \mathfrak{S} = H^{-1}(\Sigma), \mathfrak{S}_+ = H^1(\Sigma) \), and in \( n = 4 \) we have \( \mathfrak{S} = H^{-1/2}(\Sigma), \mathfrak{S}_+ = H^{1/2}(\Sigma) \).

The elements from the defined domain of the adjoint operator \( D_{\widetilde{H}_0^+} \) have singularities on \( \Sigma \):

\[
f \equiv f_0 + (2\pi)^{-1} \log r^{-1} F(t), \quad n = 3;
\]

\[
f \equiv f_0 + (4\pi^2)^{-1} r^{-1} F(t), \quad n = 4.
\]

To separate the necessary elementary extensions, one needs to impose a boundary condition on these elements. To do this, the trace operator \( i_\Sigma \) with \( H^2(\mathbb{R}^n) \) has to be extended to the compact subset \( D_0 \subset D_{\widetilde{H}_0^+} \) characterized by the affiliation \( F \in C^\infty_0(I \setminus I_0) \), i.e., the renormalization trace operator \( i_r \) needs