The Dirac equation is derived in the Foldy-Wouthuysen representation describing the interaction of spin-1/2 relativistic particles with an external electromagnetic field; it is valid in the weak-field approximation. In contrast to previous studies, this equation includes all of the order derivatives of the field potentials. The quantum-mechanical equation is obtained for the spin motion in the Foldy-Wouthuysen representation; it is consistent with the classical Bargmann-Michel-Telegdi equation.

1. Introduction

The Dirac equation describing the interaction of spin-1/2 particles with an external field supplemented with terms characterizing the anomalous magnetic moment $\mu'$ of a particle in the relativistic system of units ($\hbar = c = 1$) has the form [1, 2]

$$\left[ \gamma^\mu (\not p - e \not A) - m + i \mu' \sigma^{\mu\nu} F_{\mu\nu}/2 \right] \Psi = 0,$$

where $\gamma^\mu = (\gamma^0, \gamma^i)$ are Dirac matrices; $A^\mu = (\Phi, \vec{A})$ and $p^\mu = (\not p, \vec{p})$ are, respectively, the four-dimensional potential of an external field and the momentum of a particle; $\vec{p} = -i \nabla$, $F_{\mu\nu}$ is the electromagnetic field tensor, $\sigma^{\mu\nu} = (\gamma^\mu \gamma^\nu - \delta^{\mu\nu} \gamma^5)/2$, $m$ is the rest mass. If the external field is constant, the operator $\varepsilon \equiv i \partial_{\vec{p}}$ can be replaced by the total energy of the stationary states of a particle $E$. The four-component bi-spinor $\Psi$ in standard representation has the form $\Psi = (\phi, \chi)$, where $\phi$ and $\chi$ are two-component spinors of different parity. The Hamiltonian in the Dirac representation is [1, 2]

$$H_D = \vec{\alpha} (\vec{p} - e \vec{A}) + \beta m + e \Phi + \mu' (\vec{\sigma} \vec{H} + i \vec{\sigma} \vec{E}),$$

where $\vec{\alpha} = \gamma^0 \vec{\gamma}$, $\beta = \gamma^0$, $\vec{E}$ and $\vec{H}$ are electric and magnetic vectors of the external field strength, and $\vec{\sigma}$ are Pauli matrices. The Hamiltonian $H_D$ is not even. An important problem is to rewrite Eq. (2) in the Foldy-Wouthuysen (FW) representation when the Hamiltonian is even and the equations for the two spinors are separated. This transformation has been performed using the FW method by representing the Hamiltonian as a series in powers of $c^{-2}$ [3]. In this case, the resulting compact expressions are obtained for the Hamiltonian only in the nonrelativistic case when this series is rapidly convergent. Since the Dirac spinors $\phi$ and $\chi$ are in one-to-one correspondence, one of them can completely describe the state of a particle. Separation of the variables for the two spinors in the FW representation allows us to use the Dirac equation for the two-component Hamiltonian $H$:

$$i \frac{\partial \psi}{\partial t} \equiv \varepsilon \psi = H \psi,$$

where the operator $H$ acts on the two-component wave function $\psi$. Transition to Eq. (3) was also realized by the operator method in [4], in which the Hamilton operator was represented as a series in powers of $c^{-2}$.  

---

More general results were obtained by a modification of the FW method in [5] and especially in [2], where relativistic expressions were derived for the Hamiltonian in the FW representation. The Hamiltonian in [2], however, was obtained when the second- and higher-order derivatives of potentials, i.e., terms that are not small, were neglected. The authors of [2] employed the weak-field approximation, when the interaction energy is small, as compared to the total energy of a particle including its rest mass ($|W_{\text{int}}| \ll E$).

Equations (1)–(3) hold only within the one-particle approximation when the radiative corrections are not calculated in a consistent way but are phenomenologically taken into account by including extra terms in the Dirac equation (see [6]). The one-particle description is feasible even for ultra-relativistic particles if the external field is so weak that the probability of pair production or losses for bremsstrahlung can be neglected for a given particle energy. The applicability range of that description is quite large and, in particular, it includes the interaction of relativistic particles with the medium and relativistic particle scattering. The FW representation possesses certain advantages: separation of spinors in this representation somewhat simplifies the Dirac equation and the form of the operators; specifically, the polarization of particles in this representation is described by the operator $\Pi$ [1].

In this paper, through further development of the operator method, we derive a relativistic expression in the FW representation for the Hamiltonian describing the interaction of particles of an arbitrary energy with an electromagnetic field. As in [2], we take advantage of the weak-field approximation, but in contrast, we consider the derivatives of all orders of the potentials. The resulting additional terms are, generally speaking, not small, and they provide a complete description of the state of a Dirac particle with an anomalous magnetic moment in a sufficiently weak electromagnetic field.

2. Two-component Hamilton operator and equation for a two-component wave function

Equation (1) can be separated in two for the spinors $\phi$ and $\chi$:

$$
\begin{align*}
\varepsilon \phi &= \left[ m + e\Phi - \mu'\sigma \bar{H} + (\bar{\sigma}\pi + i\mu'\sigma \bar{E})U^{-1}(\bar{\sigma}\pi - i\mu'\sigma \bar{E}) \right] \phi, \\
\varepsilon \chi &= \left[ -m + e\Phi + \mu'\sigma \bar{H} + (\bar{\sigma}\pi - i\mu'\sigma \bar{E})T^{-1}(\bar{\sigma}\pi + i\mu'\sigma \bar{E}) \right] \chi,
\end{align*}
$$

which are connected by the equations

$$
\begin{align*}
\phi &= T^{-1}(\bar{\sigma}\pi + i\mu'\sigma \bar{E})\chi = R\chi, \\
\chi &= U^{-1}(\bar{\sigma}\pi - i\mu'\sigma \bar{E})\phi = Q\phi, \\
U &= \varepsilon + m - e\Phi - \mu'\sigma \bar{H}, \\
T &= \varepsilon - m - e\Phi + \mu'\sigma \bar{H}, \\
\bar{\pi} &= \bar{p} - e\bar{A}.
\end{align*}
$$

We will use the following formal notation of the first equation of (4) (the field strengths are expressed in terms of commutators of the operators $A^\mu$ and $\bar{p}$):

$$
\varepsilon \phi = H_0(\varepsilon, A^\mu, \bar{p})\phi \equiv H_0\phi,
$$

where $H_0$ is a Hermitian operator, as follows from (4). It is a two-component operator since it acts on a two-component wave function; however, $H_0$ is not Hamiltonian, since it depends on $E$. For stationary states, $\varepsilon \phi = E\phi$ and Eq. (6) expresses $E$ through $A^\mu$, $\bar{p}$ implicitly. To determine the Hamilton operator in an explicit form, it is necessary to construct an equation of type (3), where the operator $H \equiv H(A^\mu, \bar{p})$ and does not contain $E$. A method of solving this problem is the operator method developed by Berestetskii and Landau (see [7]), which introduces a two-component spinor $\psi$ satisfying Eq. (3) and transforms it into $\phi$ in the nonrelativistic case. The wave functions $\psi$, like the Dirac bispinor $\Psi$, are normalized to unity:

$$
\int \psi^\dagger \psi \, dV = 1, \quad \int \Psi^\dagger \Psi \, dV = \int (\phi^\dagger \phi + \chi^\dagger \chi) \, dV = \int \phi^\dagger (1 + Q^\dagger Q) \phi \, dV = 1.
$$

The last of these equalities can be written in the form

$$
\int \phi^\dagger P^\dagger \phi \, dV = 1,
$$