The existence of a so-called "phantom" scalar field in some Riemannian space $V_5$, i.e., a field in which the effective energy momentum tensor $T^{(5)}$ vanishes in the five-dimensional Kaluza—Klein theory, is investigated by means of the integrability conditions for relations of the form $\Phi_{\mu,\nu} = k\Phi_{,\mu,\nu} + bg_{\mu\nu}$ found in [6]. Phantom fields are found in homogeneous isotropic cosmological models.

1. PROJECTIVE FORMALISM

Difficulties associated with the interpretation of the geometric scalar field in the five-dimensional Kaluza—Klein theory have existed since the birth of multi-dimensional theories and were among the factors responsible for the lessening of interest in multidimensional theories in the 1920s and 1930s [2, 3]. The earliest studies on multidimensionality employed the following coupling condition: $G_{55} = \text{const.}$ All subsequent analogs of multidimensional theories may be classified on the basis of the status assumed in the particular theory by the geometric scalar field. A survey and brief description of different branches of multidimensional theories may be found in [3].

We will employ the mathematical apparatus of $1+4$ splitting of $V_5$ to study the four-dimensional effective properties of the five-dimensional Riemannian manifold $V_5$; the technique was first introduced in [2]. Its underlying idea, in the monadal formalism of general relativity theory, is to make use of a certain normalized vector field as a means of identifying the appropriate direction and the subspace orthogonal to it [4]. Once the field of the five-dimensional monad $\lambda_A$ such that $\lambda_A\lambda^A = -1$ has been specified in some way, the metric tensor of an arbitrary five-dimensional Riemannian manifold may be represented in the following form:

$$G_{AB} = \tilde{g}_{AB} - \lambda_A\lambda_B.$$  \hspace{1cm} (1)

Next, by analogy with the four-dimensional monadal formalism, an algebra is constructed by means of which four-dimensional projections of five-dimensional tensors may be determined. In addition, physico-geometric tensors corresponding to the strengths of the geometrized vector and scalar fields and the monadal differentiation operators are introduced. The latter are then used to write down the five-dimensional Einstein equations in projective form. The general relations and equations were presented in two previous articles [2, 4]. Let us briefly enumerate the basic assumptions of the five-dimensional monadal method. These consist in:

(a) gauging of the monad $\lambda_A$ is analogous to chronometric gauging in the four-dimensional monadal formalism:

$$\lambda^A = \frac{G^A_5}{\sqrt{-G_{55}}} \to \lambda_A = \frac{G_{AS_5}}{\sqrt{-G_{55}}}.$$  \hspace{1cm} (2)

(b) the Riemann space $V_5$ admits the space-like Killing vector which, in the present case, may always be directed along the tangent to the coordinate line $x^5$. The generalized frame determined by the monadal vector $\lambda_A$ in the gauge (2) will then

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No such condition was imposed in the pioneering study of Kaluza [1].
be a Killing frame, and the four-dimensional projection of the five-dimensional analog of the rate of deformation tensor will be zero:

\[(1/2q) \frac{\partial \tilde{e}_\mu}{\partial x^5} = 0,\]  

where

\[\psi = \sqrt{-G_{55}} \]  

is a geometric scalar field;

(c) the five-dimensional coordinate transformations contract to the following:

\[x'^\nu = x'^\nu(x^0, x^1, x^2, x^3),\]

\[x'^5 = x^5 + f(x^0, x^1, x^2, x^3)\]

These transformations, on the one hand, preserve the condition of cylindricity (3) and the four-dimensional covariance of the metric \(g_{\mu\nu}\) from (1), the scalar field (4), and the vector field

\[A_\mu = \frac{e^2}{2 \sqrt{\psi}} \lambda_\mu\]

On the other hand, they constitute general-coordinate four-dimensional transformations and gauge transformations of the vector potential \(A_\mu\);

(d) the four-dimensional metric \(\tilde{g}_{\mu\nu}\) is defined to within a conformal transformation of the original metric from (1):

\[\tilde{g}_{\mu\nu} = F(\psi)g_{\mu\nu},\]

where \(F(\psi)\) is an arbitrary function of the scalar field (4). In the present article we are limiting ourselves to the class of functions of the form

\[F(\psi) = \psi^n,\]

where \(n\) is any real number;

(e) the further analysis will assume that the strength tensor of the vector field (7) is zero:

\[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0;\]

(\(f\) the multi-dimensional Einstein equations are assumed to have the following form:

\[5R_{AB} - \frac{1}{2} G_{AB} = \kappa T_{AB},\]

where \(5R_{AB}\) is the five-dimensional Ricci tensor, \(5R\) the five-dimensional scalar curvature, and \(5T_{AB}\) the energy momentum tensor of five-dimensional matter. Equations (11) may be obtained by variation (with respect to \(G_{AB}\)) of operations of the following form:

\[S = \int 5R \sqrt{G} d^5x + S_{mat},\]

where \(\sqrt{G}R\) is the hyperdensity of the Lagrangian of the five-dimensional gravitational field and \(S_{mat}\) is the action of five-dimensional matter. The constraints (b), (c), and (d) are imposed following the variation procedure.

With all the steps of the foregoing discussion in mind, it is clear that Eqs. (11) possess the following projections: