CLASSICAL FORMULATION OF QUANTUM MECHANICS

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Abstract

The concept of quantum state is given in terms of classical probability for position in squeezed and rotated classical reference frames in phase space. Stationary states and energy levels of the quantum system are obtained in a classical formulation of quantum mechanics. The positive probability density of the harmonic oscillator position is obtained by solving a new eigenvalue equation of standard quantum mechanics instead of the Schrödinger equation. The orthogonality and completeness relations are found for the eigendistributions.

1. Introduction

Standard quantum mechanics is based on concept of a complex wave function which satisfies the Schrödinger equation [1]. Attempts to give classical-like interpretations of the wave function have been made in [2–4]. It turned out that a new formulation of standard quantum mechanics may be given in terms of classical probabilities for a position [5, 6] based on a symplectic tomography scheme [7, 8]. Recently, the energy levels of a harmonic oscillator were discussed in a classical formulation of quantum mechanics [9], as were the transition probabilities between the levels. The use of marginal distributions for a homodyne observable [10, 11] to describe quantum states is based on some relations of the density matrix to the characteristic functions for the observable [12]. A marginal distribution for the position in the ensemble of shifted, rotated, and scaled reference frames in classical phase space of the system under study has been introduced [7]. It was shown that this marginal distribution determines the quantum state, since the Wigner function is given by a Fourier component of the marginal distribution. An invariant form for the connection of the marginal distribution with a density matrix was found [8], and the approach was extended to a system with several degrees of freedom.

A classical formulation of quantum evolution was suggested and for the marginal distribution a new quantum evolution equation was found [5], which is an alternative to the time-dependent Schrödinger equation. This equation gives a classical-like description of quantum evolution in terms of a positive normalized distribution containing complete information about the state of the system. Examples of free motion and some excited states of the harmonic oscillator were also considered [5, 6]. The evolution of even and odd coherent states [13] of a particle in a Paul trap was investigated [14] in a classical-like description [5–7]. The even and odd coherent states of a trapped ion were discussed in recent papers [15, 16].

The aim of this work is to discuss the concept of quantum state in a new formulation of quantum mechanics. We review the classical-like description of transition probabilities between stationary states (energy levels) of quantum systems and obtain analogs of the orthogonality and the completeness relations. We show that, if the evolution equation describing the dynamics of a quantum system is determined by the imaginary part of the system potential energy considered as a function of the complex coordinate, the energy states of the system are determined by the real part of the potential energy. The energy levels of the harmonic oscillator are rederived in classical-like alternatives to the Schrödinger evolution and stationary equations. A new type of eigenvalue problem is formulated for the positive and normalized marginal distributions.
2. **New Concept of Quantum State**

It was shown [7] that for the generic linear combination of quadratures which is a measurable observable \( (\hbar = 1) \)
\[
\hat{X} = \mu \hat{q} + \nu \hat{p} + \delta,
\]  
where \( \hat{q} \) and \( \hat{p} \) are the position and momentum, respectively, the marginal distribution \( w(X, \mu, \nu, \delta) \) (normalized with respect to the \( X \) variable), depending upon three extra real parameters \( \mu, \nu, \delta \), is related to the state of the quantum system expressed in terms of its Wigner function \( W(q, p) \) as follows
\[
w(X, \mu, \nu, \delta) = \int \exp \left[ -i k (X - \mu q - \nu p - \delta) \right] W(q, p) \frac{dk\, dq\, dp}{(2\pi)^2}.
\]  
As it follows from this formula, the marginal distribution depends on the difference of the variables \( X - \delta \). The physical meaning of the parameters \( \mu, \nu, \delta \) is that they describe the ensemble of shifted, rotated, and scaled reference frames in which the position \( X \) is measured. Formula (2) can be inverted and the Wigner function of the state can be expressed in terms of the marginal distribution [7]
\[
W(q, p) = \frac{1}{2 \pi} \int \exp \left( isX \right) \, w_F(X, sq, sp, s),
\]  
where \( w_F(X, a, b, s) \) is the Fourier component of the marginal distribution (2) taken with respect to the parameters \( \mu, \nu, \delta \), namely,
\[
w_F(X, a, b, s) = \int w(X, \mu, \nu, \delta) \exp \left[ -i (a \mu + b \nu + s \delta) \right] \frac{d\mu\, d\nu\, d\delta}{(2\pi)^3}.
\]  
Since the Wigner function determines completely the quantum state of a system and, on the other hand, this function itself is completely determined by the marginal distribution, one can understand the notion of the quantum state in terms of the classical marginal distribution for squeezed and rotated quadrature. So, we say that the quantum state is given if the position probability distribution \( w(X, \mu, \nu, \delta) \) in the ensemble of rotated, squeezed, and shifted reference frames in classical phase space is given.

3. **Energy Levels and Quantum Dynamics**

As was shown [5], for systems with the Hamiltonians
\[
H = \frac{\hat{p}^2}{2} + V(\hat{q}),
\]  
the marginal distribution satisfies the quantum time-evolution equation, which is the integral equation determined by the imaginary part of the potential energy considered as a function of the complex coordinate. The evolution equation is
\[
\dot{w} - \mu \frac{\partial}{\partial \nu} w - i \left[ \frac{1}{2} \frac{\partial}{\partial q} \frac{\partial}{\partial \mu} - \nu \frac{\partial}{2 \partial X} \right] - V \left( \frac{1}{2} \frac{\partial}{\partial \mu} + \frac{1}{2} \frac{\partial}{\partial X} \right) \right] w = 0.
\]  
The measurable position is a cyclic variable for the evolution equation.

Let us rewrite the Schrödinger equation for the stationary state density matrix \( \rho_E \) of a quantum system with Hamiltonian (5)
\[
H \rho_E = \rho_E H = E \rho_E
\]