SOLVABILITY OF NONLINEAR STATIONARY TRANSFER EQUATION

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On the basis of a theorem on the global solvability of abstract equations that generalizes earlier results of the author, theorems on the solvability of the nonlinear stationary one-velocity transfer equation for the general and periodic cases are obtained.

1. GLOBAL SOLVABILITY OF EQUATIONS IN QUASISEMINETRIC SPACES

1.1. We recall the definitions of [2] (p. 49). Let X be a set, and \( \rho: X \times X \to [0, + \infty] \) be a function that satisfies the conditions

\[
(x \in X) \Rightarrow \rho(x, x) = 0, \quad (x, x', x'' \in X) \Rightarrow \rho(x, x') \leq \rho(x, x'') + \rho(x'', x').
\]

Then we call \((X, \rho)\) as quasisemimetric space (QSMP), and \(\rho\) a quasisemimetric. If

\[
(x, x' \in X, \rho(x, x') = 0) \Rightarrow x = x',
\]

then we call \((X, \rho)\) a quasimetric space, and \(\rho\) a quasimetric. Let \((X, \rho)\) be a QSMP. We shall say that a sequence \(x_n \in X\) is monotonically fundamental if

\[
\forall \varepsilon > 0 \ \exists N : (m > n > N) \Rightarrow \rho(x_n, x_m) < \varepsilon.
\]

We say that a sequence \(x_n \in X\) is counterconvergent to a point \(x \in X\), and \(x\) is the counterlimit of the sequence \(x_n\) (denoted

\[
x = \lim_{n \to \infty} x_n,
\]

if

\[
\forall \varepsilon > 0 \ \exists N : (n > N) \Rightarrow \rho(x_n, x) < \varepsilon.
\]

We shall say that a set \(W\) in a QSMP is monotonically complete if each monotonically fundamental sequence \(x_n \in W\) counterconverges to some element \(x \in W\). We shall say that a set \(W\) in the QSMP \(X\) is counterclosed if

\[
\{x_n\} \subseteq W, \quad x \in X, \quad x = \lim_{n \to \infty} x_n \Rightarrow x \in W.
\]

1.2. Example. Let \(X\) be a linear space, \(K\) be a convex cone in \(X\) with vertex at the point \(0 \in K\), and \(x \mapsto \|x\|\) be a nonnegative sublinear (i.e., convex and positively homogeneous) function on \(K\). In this case, we shall say that \(K\) is a seminormed cone, and \(\|\cdot\|\) is a seminorm on \(K\). For \(x, x' \in K\) we set \(\rho(x, x') = \|x' - x\|\) for \(x' - x \in K\) and \(\rho(x, x') = + \infty\) for \(x' - x \notin K\). Then \((X, \rho)\) is a QSMP. If \(\|x\| = 0\) only for \(x = 0\), then we call \(K\) a normed cone, and \(\|\cdot\|\) a norm on \(K\) (examples of equations in normed cones are given in [4, 3] (p. 108, 111).

1.3. THEOREM (cf. [2], p. 50). Let \((X, \rho)\) be a monotonically complete QSMP, \(Y\) be a normed cone, \(x_0 \in D \subset X\), and the mapping \(f: D \to Y\) have a graph that is counterclosed (in the natural sense) in \(X \times Y\). For \(x \in D\), we set

\[
k(f, x) = \sup \{k \geq 0 \mid \forall y \in Y, \|y\| = 1, \forall \delta > 0 \exists \xi \in [0, \delta] \exists x' \in D, \rho(x, x') \leq \xi : f(x') = f(x) + k\xi y\}.
\]

Let

\[
k(f, x) \geq b(\rho(x_0, x)), \quad x \in D,
\]

where the function \(b: [0, + \infty] \to \mathbb{R}\) is nonnegative, monotonically nonincreasing, and such that
\[ \omega = \int_0^\infty b(r)dr \in [0, +\infty]. \] 

Then
\[ \{ y \in Y \mid \| y - f(x_0) \| < \omega \} \subset f(D). \]

Proof. It can be assumed that \( f(x_0) = 0 \). Let \( y \in Y \) and \( 0 < \| y \| < \omega \). Then \( \| y \| < \theta \omega \) for some \( \theta \in [0,1] \). Let
\[ r_0 = \sup\{ r > 0 \mid b(r) > 0 \}, \quad u(r) = \theta \int_0^r b(s)ds \]
for \( r \in [0, r_1] \) and \( v = u^{-1} \). Then \( v \) is defined on \( [0, \theta \omega] \), increases monotonically, and is convex. Consider the set \( \mathcal{U} \) of all pairs \( (\Delta(g), g) \) consisting of a set \( \Delta \subset [0, \| y \|] \) such that \( 0 \in \Delta(g) \) and a function \( g : \Delta(g) \to D \) such that
\[ g(0) = x_0; \quad (t \in \Delta(g)) \Rightarrow f(g(t)) = t\| y \|^{-1}y; \quad (t, t' \in \Delta(g), t \leq t') \Rightarrow \rho(g(t), g(t')) \leq v(t') - v(t). \] 

(1.4)

Obviously, \( \mathcal{U} \neq \emptyset \). We order the set of graphs of such functions by inclusion, choose in it a maximal chain, and consider the union of all graphs in this chain. We then obtain a set \( \Delta(g) \subset [0, \| y \|] \) and function \( g : \Delta(g) \to D \) satisfying (1.4). Let
\[ T = \sup \Delta(g), \quad t_n \in \Delta(g) \quad \text{and} \quad t_n \nearrow T. \]

Then
\[ \rho(\bar{g}(t_n), \bar{g}(t_m)) \leq v(t_m) - v(t_n) \to 0 \]
as \( m, n \to \infty, \ m \geq n \) by virtue of the uniform continuity of \( v \) on \( [0, \| y \|] \). We set \( x_n = \bar{g}(t_n) \). Then by virtue of the monotonic completeness of \( X \) there exists
\[ \lim_{n \to \infty} x_n = x \in X, \]
and
\[ f(x_n) = f(\bar{g}(t_n)) = t_n\| y \|^{-1}y \]
counterconverges to \( T\| y \|^{-1}y \) in \( Y \). Because the graph of \( f \) is counterclosed in \( X \times Y \), the relations \( x \in D \) and \( f(x) = T\| y \|^{-1}y \) hold. If \( T \notin \Delta(g) \), then we set \( \bar{g}(t) = \bar{g}(t) \) for \( t \in \Delta(g) \) and \( \bar{g}(T) = x \). Then for \( t \in \Delta(g) \)
\[ \rho(\bar{g}(t), \bar{g}(T)) = \rho(\bar{g}(t), x) \leq \lim_{n \to \infty} \rho(\bar{g}(t), \bar{g}(t_n)) \leq \lim_{n \to \infty} [v(t_n) - v(t)] = v(T) - v(t), \]
since
\[ \rho(\bar{g}(t), x) \leq \rho(\bar{g}(t), \bar{g}(t_n)) + \rho(\bar{g}(t_n), x) \]
and \( \rho(\bar{g}(t_n), x) \to 0 \) as \( n \to \infty \). In this case, \( g \) is a continuation of the function \( \bar{g} \) that satisfies (1.4), and this contradicts the maximality of the pair \( (\Delta(g), g) \). Therefore, \( T \in \Delta(g) \). Suppose \( T \notin \| y \| \). Since
\[ \rho(x_0, x) = \rho(\bar{g}(0), \bar{g}(T)) \leq v(T) - v(0) = v(T) \]
and, therefore,
\[ \theta b(\rho(x_0, x)) \geq \theta b(v(T)) \geq \lim_{r \to v(T)+0} \theta b(r) = u_{np}(v(T)) = [u_{np}(T)]^{-1} > 0, \]
(1.5)
it follows by virtue of (1.2) that
\[ k(f, x) > \theta b(\rho(x_0, x)). \]
In this case, it follows from (1.1) (for the point \( x \)) that
\[ \exists \ k > \theta b(\rho(x_0, x)) \forall \delta > 0 \exists \xi \in [0, \delta] \exists x' \in D, \quad \rho(x, x') \leq \xi : f(x') = (T + k\xi)\| y \|^{-1}y. \] 
(1.6)

Since \( T \notin \| y \| \), it follows that
\[ \exists \delta > 0 : T + k\delta < \| y \|. \]
For this \( \delta \), we find from (1.6) the corresponding \( \xi \) and \( x' \). We set \( \lambda = k\xi \), \( \bar{g}(t) = \bar{g}(t) \) for \( t \in \Delta(g) \) and \( \bar{g}(T + \lambda) = x' \). Then \( T + \lambda < \| y \| \) and for \( t \in \Delta(g) \) we obtain by virtue of (1.4) and (1.5)