ON THE ASYMPTOTIC EVOLUTION OF A LOCALIZED PERTURBATION OF THE ONE-DIMENSIONAL LANDAU–LIFSHITS EQUATION WITH UNIAXIAL ANISOTROPY

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The Witham method is applied to analyze the modulation instability of plane waves in a one-dimensional ferromagnet described by the Landau–Lifshits equation with uniaxial anisotropy. It is shown that the instability results in the formation of a domain structure in the system.

1. Introduction

As is known, the homogeneously magnetized state of a ferromagnetic material in the absence of a sufficiently strong magnetic field is unstable. It disintegrates into domains, i.e., regions of homogeneous magnetization, but where the magnetization vectors in neighboring regions are oppositely directed. This state is more energetically favorable due to the reduction in the total energy of the magnet and the fact that it can be found from the minimum condition for its free energy. The shape and size of the equilibrium domains depend on the geometry of the sample.

In quasi-homogeneous ferromagnets [1–3], there is no such determining effect for the boundary. However, the appearance of a domain structure is also possible but, here, it is due to the modulation instability of the system. The simplest nonlinear formations appearing in one-dimensional ferromagnets can be studied theoretically using elementary integration methods for the corresponding Landau–Lifshits equation describing the magnetization dynamics (see, e.g., [4]). Unfortunately, the particular solutions found in this way are insufficient for the theoretical description of more complex situations. As an example, consider the following problem. Let a homogeneously magnetized sample be created at the initial instant of time (e.g., by switching off the initially applied magnetic field). As a result of the instability of this state, a perturbation develops and becomes a region of inhomogeneous magnetization propagating through the magnet. This process is similar to the self-modulation of an unstable wave, which has long been known in the physics of nonlinear waves (e.g., see [5]). This phenomenon was studied in [6], using the Witham method [7]. An analogous approach was used in [8] to describe the evolution of an inhomogeneity initially having the form of an abrupt interface between two homogeneously magnetized regions in the Heisenberg isotropic magnet. In the present paper, we extend the method of [6, 8] to the more realistic case (compared to [8]) of a homogeneous ferromagnet with “easy axis”-type anisotropic energy. Note that although all three of the above-mentioned problems within the framework of the Witham method lead to the same system of Witham equations for the so-called Riemann invariants, each individual case is characterized by its own relationship between these invariants and the “physical variables.”

The Witham method [7] is based on the assumption that the region of inhomogeneity can be represented as a modulated periodic wave (depending on an appropriate number of parameters) for which the typical parameter-variation scale is much greater than the wavelength. This condition permits the equations of motion to be averaged with respect to the fast oscillations of the wave in order to derive modulation...
equations describing the evolution of the region of inhomogeneity. A general approach to finding periodic solutions to the Landau–Lifshits equations, based on the “finite-zone” integration method, was suggested in [9, 10]. The effective form we need for the periodic solution of the Landau–Lifshits equation in the case of a uniaxial ferromagnet was found in [11]. In the present paper, we derive the corresponding Witham equation (Sec. 2) and construct a solution describing the evolution of an inhomogeneous region decomposing into domains (Sec. 3).

2. Periodic solutions and the Witham equations

A uniaxial one-dimensional ferromagnet is described by the Landau–Lifshits equation

\[
\frac{\partial \mathbf{M}}{\partial t} = \left[ \mathbf{M}, \frac{\partial^2 \mathbf{M}}{\partial z^2} \right] + J(\mathbf{M}, \mathbf{n})[\mathbf{M}, \mathbf{n}],
\]

(1)

where \( \mathbf{M}(z, t) \) is the local magnetization, \( J > 0 \) is the anisotropy constant, and \( \mathbf{n} \) is the unit vector along the easy magnetization axis \( z \) coinciding with the coordinate axis in the direction of wave propagation. The vector \( \mathbf{M} = (M_1, M_2, M_3) \) is normalized by the condition of unit length

\[
\mathbf{M}^2 = 1.
\]

(2)

The periodic solution found in [11] is defined on the elliptic curve

\[
y^2 = P(\lambda) = \prod_{i=1}^{4} (\lambda - \lambda_i) = \lambda^4 - s_1 \lambda^3 + s_2 \lambda^2 - s_3 \lambda + s_4
\]

(3)

and plays a major role in the finite-zone integration theory. Here \( s_1, s_2, s_3, \) and \( s_4 \) are integrals of the equations of motion in the periodic solution. The zeros \( \lambda_i \) of the polynomial \( P(\lambda) \) split into two complex conjugate pairs,

\[
\lambda_1 = \alpha + i\gamma, \quad \lambda_2 = \beta + i\delta, \quad \lambda_3 = \alpha - i\gamma, \quad \lambda_4 = \beta - i\delta.
\]

(4)

These pairs turn out to be the Riemann invariants whose evolution is governed by the Witham equations. However, it is convenient to express the solution itself via the four real parameters \( \nu_i, \ i = 1, 2, 3, 4, \) related to \( \lambda_i \) by some complex expressions. We present only one of them here,

\[
\nu_1 = (4f_1J)^{-1} \left[ (\lambda_1 - \lambda_3) (\lambda'_2 - \lambda'_4) + (\lambda_2 - \lambda_4) (\lambda'_1 - \lambda'_3) \right]^{-1} \times
\]

\[
\times \left\{ (\lambda_1 - \lambda_3) [2(\lambda_1 + \lambda_3)(\lambda'_2 - \lambda'_4)J + (\lambda_2 \lambda'_4 - \lambda_4 \lambda'_2)( (\lambda_1 + \lambda_3)^2 - (\lambda'_1 - \lambda'_3)^2 )] + (\lambda_2 - \lambda_4) [2(\lambda_2 + \lambda_4)(\lambda'_1 - \lambda'_3)J + (\lambda_1 \lambda'_3 - \lambda_3 \lambda'_1)( (\lambda_2 + \lambda_4)^2 - (\lambda'_2 - \lambda'_4)^2 )] \right\},
\]

(5)

where

\[
\lambda'_i = \sqrt{\lambda_i^2 + J},
\]

(6)

\[
f_1 = \frac{1}{2J} (\lambda'_1 \lambda'_2 \lambda'_3 \lambda'_4 - J^2 + s_2 J - s_4).
\]

(7)

Parameter \( \nu_2 (\nu_3) \) is obtained from \( \nu_1 \) by the change of indices \( 3 \leftrightarrow 4 \ (3 \leftrightarrow 2) \) and \( \nu_4 \) is found by the formula

\[
\nu_4 = \frac{s_1 J - s_4}{f_1 J} - (\nu_1 + \nu_2 + \nu_3).
\]

(8)