THREE-LOOP CALCULATION OF THE
ANOMALOUS FIELD DIMENSION IN THE FULL
FOUR-FERMION U_N-SYMMETRIC MODEL

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We present a three-loop calculation for the contribution of the anomalous dimension \( \gamma_\psi \) in the RG function for the full \( U_N \)-symmetric model using dimensional regularization with \( d = 2 + \epsilon \). This model contains an infinite number of independent four-fermion interactions and couplings \( g_n \). In the MS scheme, the three-loop contribution in \( \gamma_\psi \) depends on all of the charges. In the symmetric subtraction scheme for \( \epsilon = 0 \), only the dependence on the three lower charges \( g_{0,1,2} \) remains. The results of a three-loop calculation of \( \gamma_\psi \) for the Thirring model and for the Gross-Neveu model are discussed; they seem to contradict the equivalence of all four-fermion interactions for \( d = 2, N = 1 \). It is shown that this equivalence appears only for the symmetric subtraction scheme. The precise meaning of \((2 + \epsilon)\)-expansions of the critical dimensions is discussed for when the multiplication renormalizability is absent in the Gross-Neveu model.

1. Introduction

The full \( U_N \)-symmetric Euclidean four-fermion (4F) model \([1, 2]\) describes a system of \( N \) species of \( d \)-dimensional Euclidean Dirac spinors \( \phi \equiv \{ \psi_\alpha, \bar{\psi}_\alpha, \alpha = 1, \ldots, N \} \) with a general four-fermion interaction including all local vertices

\[
V_n = \frac{1}{2} \bar{\psi} \gamma^{(n)}_A \gamma A \psi, \tag{1}
\]

in which \( \gamma^{(n)}_A \) is an anti-symmetrized product of \( \gamma \)-matrices:

\[
\gamma^{(n)}_A \equiv A_s[\gamma_{i_1}, \ldots, \gamma_{i_n}], \quad A \equiv \{i_1, \ldots, i_n\}. \tag{2}
\]

When the dimension \( d \) is a half-integer, the values (2) with any number of factors must be considered as independent structures different from zero. Any product of \( \gamma \)-matrices (for \( d \)-dimensions, it would be more accurate to say "of \( \gamma \)-symbols") can be reduced to a linear combination of the magnitudes (2) with the commutation relation \( \gamma_i \gamma_k + \gamma_k \gamma_i = 2 \delta_{ik} \). We do not explicitly write the isotopic indices \( \alpha = 1, 2, \ldots, N \) for the fields \( \psi, \bar{\psi} \) in (1) and further, meaning summation with respect to it in all quadratic forms \( \bar{\psi} \psi \) (\( U_N \)-symmetry), as well as summation with respect to the repeated vector indices and multi-indices of type \( A \) from (2) in all formulas.

Thus, the nonrenormalized functional of the action of the full \( U_N \)-symmetric \( d \)-dimensional massless 4F-model is of the form \([1, 2]\)

\[
S(\phi) = \bar{\psi} \hat{\delta} \psi + \sum_{n=0}^{\infty} g_{0n} V_n, \tag{3}
\]

where \( g_0 \equiv \{ g_{0n}, n = 0, 1, 2, \ldots \} \) is an infinite set of independent basic charges. In (3) and similar formulas, integration is implied with respect to the \( d \)-dimensional coordinate \( x \). Formula (3) corresponds to the expression \( \exp S(\phi) \) without the usual minus sign in the exponent of Euclidean functional integrals.

All interactions in (1) are logarithmic for dimension \( d = 2 \) (this is the dimension for which all vertex diagrams are logarithmically divergent). We shall consider the 4F-model under the dimensional regularization...
Then, the introduction of all vertices (1) with independent charges \( g_{0n} \) is necessary as soon as we want to have a multiplicatively renormalizable theory (this is necessary, in particular, for derivation of the standard RG equations); note that the lower vertices (1) with \( n = 0, 1, 2 \) produce the higher vertices (with \( n > 2 \)) as counter-terms.

The functionals corresponding to (3) of the basic action \( S_{\text{bas}}(\phi) \) (to diagrams to which the \( R \)-operation is applied) and of the renormed action \( S_R(\phi) \) in \( d = 2 + 2\epsilon \) are of the form

\[
S_{\text{bas}}(\phi) = \bar{\psi} \partial \psi + \mu^{-2\epsilon} \sum_{n=0}^{\infty} g_n V_n,
\]

\[
S_R(\phi) = Z \bar{\psi} \partial \psi + \mu^{-2\epsilon} \sum_{n=0}^{\infty} Z_n g_n V_n,
\]

where \( \mu \) is the renormalization mass, \( Z \) are renormalization constants, and \( g = \{g_n, n = 0, 1, 2, \ldots \} \) is an infinite set of dimensionless renormalization charges. The action (5) is obtained from (3) by the standard multiplicative renormalization of fields and parameters:

\[
S_R(\phi; g, \mu) = S(Z \phi; g_0(g, \mu)).
\]

\[
g_{0k} = \mu^{-2\epsilon} g_k Z_{g_k},
\]

\[
Z_\phi \equiv Z_\psi = Z_\psi = Z^{1/2}, \quad Z_{g_k} = Z_k Z^{-2}.
\]

Therefore, the renormed 1-irreducible Green's functions \( \Gamma_{nR} \) with \( n \) fields \( \phi \equiv \bar{\psi}, \psi \) satisfy the standard RG equation

\[
[\partial_{\text{RG}} - n\gamma_\psi] \Gamma_{nR} = 0, \quad \partial_{\text{RG}} = \mu \partial_\mu + \sum_{k=0}^{\infty} \beta_k \partial_{g_k}.
\]

Here \( \beta \)-functions of the charges \( g_k \) and the anomalous field dimension \( \gamma_\psi \) (they are all called RG-functions) are defined by the relations

\[
\gamma_i = \tilde{D}_\mu \log Z_i, \quad \beta_k = \tilde{D}_\mu g_k = g_k(2\epsilon - \gamma_k),
\]

where \( \tilde{D}_\mu = D_{\text{RG}} = \mu \partial_\mu \) for fixed \( g_{0k} \).

In the MS renormalization scheme, not all anomalous dimensions \( \gamma \) depend on the parameter \( \epsilon \), which is included only in the first (loop-less) contributions in \( \beta \)-functions (8).

For the integer dimension \( d = 2 \), only the three lower vertices (1) with \( n = 0, 1, 2 \) (\( S, V, P \) in another notation) are different from zero, since when we carry out regularization and renorming for the dimension \( d = 2 \), the corresponding renormed theory proves to be a three-charge one. Under the dimensional regularization with a standard MS renormalization, the Green's functions \( \Gamma_{nR} \) depend on an infinite number of charges \( g = \{g_n\} \) of the model (5) even after transition to the limit \( \epsilon = 0 \). This would seem to contradict the general idea of the physical equivalence of any two renormed theories that are produced by different regularization and renormalization procedures from the one nonrenormed. The physical equivalence means that two systems of the renormed Green's functions are connected by some ultraviolet (UV)-finite multiplicative renormalization of fields and parameters. This is possible only in the case of the same number of independent variables. But actually, there is no contradiction here. In [2], it was shown that after the usual MS renormalization of model (4), one can perform a UV-finite renormalization of the field operator \( \phi_R \rightarrow \tilde{\phi}_R = Z_\phi^{-1}(g) \phi_R \) and of charges \( g \rightarrow g'(g) \), such that the Green's functions \( \Gamma'_{nR} \) of the new renormed field \( \phi'_R \), as functions of the new variables \( g' \), depend not on all new charges in the limit \( \epsilon = 0 \), but on the three lower ones \( g'_n(g), \ n = 0, 1, 2 \):

\[
\partial \Gamma'_{nR}(...; g', \mu) / \partial g'_{k|\epsilon=0} = 0 \quad \forall k > 2.
\]