RENORMALIZATION GROUP IN THE THEORY OF FULLY DEVELOPED TURBULENCE. COMPOSITE OPERATORS OF CANONICAL DIMENSION 8

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Renormalization and critical dimensions of the family of Galilean invariant scalar composite operators of canonical dimension eight are considered within the framework of the renormalization group approach to the stochastic theory of fully developed turbulence.

1. Introduction

In the stochastic formulation of the theory of isotropic developed turbulence of an incompressible viscous fluid, one considers the Navier-Stokes equation with an external random force [1,2]:

\[ \partial_t \varphi_i - \nu_0 \Delta \varphi_i + \partial_i p - f_i = 0, \quad \partial_t \equiv \partial_t + (\varphi \cdot \partial). \]  

Here \( \varphi \) is the transverse (by virtue of the incompressibility \( \partial_i \varphi_i = 0 \)) vector field of the velocity, \( p \) and \( f \) are the pressure and the transverse random force per unit mass (all these quantities depend on \( x \equiv t, x \)), \( \nu_0 \) is the viscosity coefficient, and \( \Delta \) is the Laplacian. The force \( f \) is considered to be Gaussian-distributed with a zero mean, and the correlator is

\[ \langle f_i(x) f_j(x') \rangle = \frac{1}{(2\pi)^d} \delta(t - t') \int dk P_{ij}(k) d(k) e^{ik(x-x')}, \]  

in which \( P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2} \) is the transverse projector, \( d \) is the dimension of space \( x \), and \( d(k) \) is a function of \( k \equiv |k| \) and of the model parameters. It can be chosen in the form

\[ d(k) = D_0 k^{4-d-2\epsilon} h(m/k), \quad h(0) = 1, \quad D_0 \equiv g_0 \nu_0^{\frac{3}{2}}, \]  

where \( g_0 \) is the bare coupling constant, \( h \) is an arbitrary, sufficiently smooth function, \( m \) is the inverse external scale of turbulence, \( \epsilon \geq 0 \) is the parameter analogous to \( \epsilon = 4 - d \) in the theory of critical phenomena, its logarithmic value is \( \epsilon = 0 \), and the physical region of the values is \( \epsilon > 2 \) (see details, for example, in [3]).

The goal of the theory is to check the basic principles of the Kolmogorov-Obukhov phenomenological theory [1], i.e., the independence of the correlation functions of velocity in the inertial interval from \( m \) (Kolmogorov's first hypothesis) and \( \nu_0 \) (second hypothesis), by the micromodel (1)-(3), to study infrared (IR) corrections to Kolmogorov scaling, and to calculate the detailed characteristics of the developed turbulence, including the critical dimensions and scaling functions of the basic fields and composite operators.

In [4], a quantum-field renormalization group (RG) approach was applied to Eqs. (1)-(3). Using this approach, the authors of [3] proved the independence of viscosity of the entire physical region \( \epsilon > 2 \). To study the dependence of the correlators on \( m \), and to check the first hypothesis, one must calculate the critical dimensions of the composite operators appearing in the operator expansions of various correlation functions, e.g., \( \langle \varphi \varphi \rangle \) (see [3]). On the other hand, it is of independent interest to calculate the critical dimensions of the composite operators, since some of them can be measured experimentally [5,6].
Schwinger functional equations and Ward identities, which express the Galilean invariance of the theory, make the task substantially easier and allow some exact results. For example, it becomes possible to exactly determine the critical dimension of the velocity field [4], local energy dissipation rate [7] and other operators appearing in the momentum and energy conservation laws [8], powers of the field \( \varphi^m [3,9] \), and all operators constructed from the velocity field and its time derivatives [10], and to find exact relations between their coefficients in the operator expansion of the product \( \varphi \varphi \) [11].

Also, the IR asymptotics of the equal-time velocity pair correlator (and thus of the pulsation energy spectrum) can be shown to be determined by the critical dimensions of Galilean-invariant scalar composite operators [12]. Up to the present time, only the operators of canonical dimension 4 and 6 have been studied in detail (see [7,8,10]). There are also some unproved hypotheses concerning operators of higher ranks [13]. In the present paper, we continue the investigation of [7–12] and consider invariant scalar operators from the next family, of canonical dimension 8.

Section 2 contains necessary information on the quantum-field formulation and renormalization of model (1)–(3); the details can be found in [3]. In Section 3, the problem of determining the IR asymptotics of the equal-time pair correlator is discussed and known information on the critical dimensions of invariant operators is given. In Section 4, renormalization of the composite operator of the dissipation rate squared is examined and it is shown not to be an operator of definite critical dimension, equal to the doubled dimension of the dissipation operator, as was earlier assumed [3,13]. In Section 5, using the Schwinger equations, we explicitly construct three composite operators of definite critical dimensions, the dimensions being found exactly.

### 2. Quantum-field formulation

The stochastic problem (1)–(3) is equivalent to the quantum theory of two transverse vector fields \( \Phi \equiv \varphi, \varphi' \) with the action functional [4]

\[
S(\Phi) = \frac{1}{2} \varphi' D^I \varphi' + \varphi'[-D_t \varphi + \nu_0 \Delta \varphi],
\]

where \( D_t \) is correlator (2) with function (3). Necessary summations over field indices and integrations over their arguments \( x = t, \ x \) in (4), and in similar formulas below, are implied.

For each \( F \) in model (4), one can introduce two independent canonical dimensions—the momentum \( d_F \) and the frequency \( d_F' \)—and their total \( d_F = d_F^k + 2d_F^\mu \) [7]. By definition, \( d_F^k = d_F^\mu = -1, \ d_F^t = d_F^x = 0 \), and the dimensions of other quantities are found under the assumption that all terms of action (4) are dimensionless (momentum-dimensionless and frequency-dimensionless, respectively). The dimensions are listed in the table (including values of renormalized parameters to be considered below).

<table>
<thead>
<tr>
<th>( F )</th>
<th>( \varphi )</th>
<th>( \varphi' )</th>
<th>( m, \mu, \Lambda )</th>
<th>( \nu, \nu_0 )</th>
<th>( g_0 )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_F^k )</td>
<td>-1</td>
<td>( d + 1 )</td>
<td>1</td>
<td>-2</td>
<td>2( \epsilon )</td>
<td>0</td>
</tr>
<tr>
<td>( d_F^\mu )</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( d_F )</td>
<td>1</td>
<td>( d - 1 )</td>
<td>1</td>
<td>0</td>
<td>2( \epsilon )</td>
<td>0</td>
</tr>
</tbody>
</table>

Model (4) is logarithmic (the total dimension of the coupling constant \( g_0 \) is equal to zero) at \( \epsilon = 0 \), and ultraviolet (UV) divergences have the form of poles \( \epsilon \) in the correlation functions of fields \( \Phi = \varphi, \varphi' \). They are completely removed at all \( d > 2 \) by one counterterm \( \varphi' \Delta \varphi \) [4], which is equivalent to multiplicative renormalization of the form

\[
\nu_0 = \nu Z_\nu, \quad g_0 = g \mu^{2\epsilon} Z_g, \quad Z_g = Z_\nu^{-3}.
\]

Here \( \mu \) is the renormalization mass in the scheme of minimum subtractions (MS), \( g, \nu \) are the renormalized analogs of the parameters \( g_0, \nu_0 \), and \( Z = Z(g, \epsilon, d) \) are the renormalization constants. Their relation in (5)