ALGEBRA OF SYMMETRY OPERATORS FOR AN INVARIANT NONLINEAR-TRANSPORT EQUATION

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The invariant form is obtained for a nonlinear-transport equation, and extended systems are found for nonlinear-transport equations in terms of the variables \((x, t), (e, t),\) and \((e, x)\), where \(x\) is the Euler coordinate, \(t\) is time, and \(e\) is the energy space variable. The algebra of point-symmetry operators is calculated for the invariant nonlinear-transport equation and this algebra is shown to be admitted by the extended systems of nonlinear transport.

1. Introduction

An effective method of constructing difference schemes for solving differential equations is based on the idea of conservatism proposed by A. A. Samarsky [1], which consists of the approximation of integral relations with respect to meshes of the difference network.

In [2], a numerical method is described for solving a one-dimensional quasi-linear transport equation. This method allows one to construct exact solutions of boundary-value problems with piecewise constant initial and boundary conditions, provided that the conditions of integrability hold throughout for the differential form corresponding to the initial quasi-linear differential transport equation. Similar ideas for constructing numerical methods to solve transport equations were used in [3].

In the present paper, it is shown that the proposed numerical method is connected with the consideration of the invariant form of an initial differential equation as described in [4] for invariant forms of gas dynamics. Extended (see [5]) systems of nonlinear-transport equations are examined in terms of the variables \((x, t), (e, t),\) and \((e, x)\). The approach used here to changing the independent variables is equivalent to the method of transforming quasi-linear systems with respect to the solution whose general form is expounded on p. 6 of [6]. The invariant nonlinear-transport equation is also written for these three cases of independent variables.

For the derived invariant nonlinear-transport equation, the operator algebra of point symmetries is calculated. It is shown that this algebra of operators, continued onto the initial dependent variable in the differential equation, is admitted by extended systems of nonlinear transport, in accordance with the definition of admissibility given in [7]. The results obtained here for the group analysis of the quasi-linear transport equation are evidently different from those derived by classical methods of group analysis [8].

2. Extended systems of nonlinear-transport equations

Consider the one-dimensional quasi-linear transport equation

\[
  u_t + u u_x = 0,
\]

which is associated with the integrable differential form [7]

\[
  dE = u dX - \frac{1}{2} u^2 dT,
\]
where $dX = dx$, $dT = dt$. Thus, the extended system of nonlinear-transport equations [5], in terms of the variables $(x, t)$, is of the form

$$u_t + u u_x = 0, \quad X_t = 1, \quad E_t = -\frac{1}{2} u^2.$$

(3)

The following integrable differential forms correspond to the equations of system (3):

$$\begin{align*}
\Gamma_1 &= u \, dx - \frac{1}{2} u^2 \, dt, \\
\Gamma_2 &= -X \, dt - dx \wedge dt, \\
\Gamma_3 &= X \, dx, \\
\Gamma_4 &= -T \, dt, \\
\Gamma_5 &= T \, dx - dx \wedge dt, \\
\Gamma_6 &= -E \, dt - u \, dx \wedge dt, \\
\Gamma_7 &= E \, dx + \frac{1}{2} u^2 \, dx \wedge dt, \\
\Gamma_8 &= dx, \\
\Gamma_9 &= dt.
\end{align*}$$

(4)

Now we make the change of variables in (4):

$$de = u \, dx - \frac{1}{2} u^2 \, dt \quad \iff \quad dx = \frac{1}{u} \, de + \frac{1}{2} u \, dt.$$

(5)

As a result of (5), the differential forms (4) become

$$\begin{align*}
\Omega_1 &= de, \\
\Omega_2 &= -X \, dt - \frac{1}{u} \, de \wedge dt, \\
\Omega_3 &= X \, de + \frac{1}{2} X \, ud\!t, \\
\Omega_4 &= -T \, dt, \\
\Omega_5 &= T \, de + \frac{1}{2} Tu \, dt - \frac{1}{u} \, de \wedge dt, \\
\Omega_6 &= -E \, dt - de \wedge dt, \\
\Omega_7 &= \frac{E}{u} \, de + \frac{1}{2} Eu \, dt + \frac{1}{u} \, de \wedge dt, \\
\Omega_8 &= \frac{1}{u} \, de + \frac{1}{2} u \, dt, \\
\Omega_9 &= dt.
\end{align*}$$

(6)

The differential forms (6) are integrable if the following equations are satisfied:

$$\begin{align*}
&\left(\frac{1}{u}\right)_t - \left(\frac{1}{2} \, u\right)_e = 0, \quad X_e = \frac{1}{u}, \quad \left(\frac{X}{u}\right)_t - \left(\frac{1}{2} \, X \, u\right) = 0, \\
&T_e = 0, \quad \left(\frac{T}{u}\right)_t - \left(\frac{1}{2} \, T \, u\right)_e = \frac{1}{u}, \quad E_e = 1, \\
&\left(\frac{E}{u}\right)_t - \left(\frac{1}{2} \, E \, u\right)_e = -\frac{1}{2} \, u.
\end{align*}$$

(7)

Upon equivalent transformations of some of the equations in (7), we obtain an extended system of nonlinear-transport equations in variables $(e, t)$:

$$\begin{align*}
u_e + \frac{1}{2} \, u^2 \, u_e &= 0, \quad X_e = \frac{1}{u}, \quad X_t = \frac{1}{2} \, u, \\
T_e &= 0, \quad T_t = 1, \quad E_e = 1, \quad E_t = 0.
\end{align*}$$

(8)

Proceeding in a similar manner and making the change of variables

$$de = u \, dx - \frac{1}{2} \, u^2 \, dt \quad \iff \quad dt = -\frac{2}{u^2} \, de + \frac{2}{u} \, dx$$

(9)

we arrive at the extended system of nonlinear-transport equations in the variables $(e, x)$:

$$\begin{align*}
u_x + \frac{1}{2} \, u \, u_x &= 0, \quad X_e = 0, \quad X_x = 1, \\
T_e &= -\frac{2}{u^2}, \quad T_x = \frac{2}{u}, \quad E_e = 1, \quad E_x = 0.
\end{align*}$$

(10)